CHAPTER IV

GEOMETRY OF MASSES

I. SYSTEMS OF POINTS

§ 1. Statical moments. Statical moment of a point. Let us consider an arbitrary plane Π. It divides space into two parts; we can consider one of these parts as positive, and the other as negative. Let A denote a certain material point and d its distance from the plane Π. We shall write $\sigma = +d$ or $\sigma = -d$, depending on whether A lies in the positive or negative part of space.

Denoting the mass of the point A by m, we shall call the expression

$$M_{\Pi} = m\sigma$$

the *statical moment* of the material point A with respect to the plane Π.

The statical moment of a point can therefore be a positive or negative number or zero (it is zero for every point A lying in the plane Π).

If we choose one of the coordinate planes xy, yz, zx, as the plane Π, then we shall consider as the positive part of space that part in which is found the positive part of the axis perpendicular to the chosen coordinate plane. If the point A of mass m has the coordinates x, y, z, then by the preceding convention we have:

$$M_{xy} = mz, \qquad M_{yz} = mx, \qquad M_{zx} = my,$$

where M_{xy}, M_{yz}, M_{zx} denote the corresponding statical moments of the point A with respect to the xy, yz and zx planes.

Statical moment of a system of points. A collection of material points is called a *system of points*, and the sum of the statical moments of its separate points is called the *(total) statical moment of the system of points*.

If the statical moments with respect to the plane Π of the material points of masses $m_1, m_2, \ldots, m_n$ are $m_1\sigma_1, m_2\sigma_2, \ldots, m_n\sigma_n$, respectively, then the total statical moment of the system of these points will be

$$M_\Pi = m_1\sigma_1 + m_2\sigma_2 + \ldots + m_n\sigma_n = \sum_{i=1}^{n} m_i\sigma_i.$$

If the material points of a given system of points have the coordinates $x_1, y_1, z_1,\ x_2, y_2, z_2, \ldots, x_n, y_n, z_n$ respectively, then the total statical moments of this system of points with respect to the corresponding coordinate planes are expressed by the formulae:

$$M_{xy} = m_1z_1 + m_2z_2 + \ldots + m_nz_n = \sum_{i=1}^{n} m_iz_i,$$

$$M_{yz} = m_1x_1 + m_2x_2 + \ldots + m_nx_n = \sum_{i=1}^{n} m_ix_i,$$

$$M_{zx} = m_1y_1 + m_2y_2 + \ldots + m_ny_n = \sum_{i=1}^{n} m_iy_i.$$

Statical moments are also called *moments of first order.*

§ 2. Centre of mass. Let there be given a system U of material points $m_1(x_1, y_1, z_1)$, $m_2(x_2, y_2, z_2)$, $\ldots$, $m_n(x_n, y_n, z_n)$. Let us consider a point S whose coordinates are:

$$x_0 = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n},\quad y_0 = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n},$$

$$z_0 = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{m_1 + m_2 + \ldots + m_n}. \tag{I}$$

The point S is called the *centre of mass* or the *centre of gravity* of the given system of points U.

The sum of the masses of the individual points (appearing in the denominators of the fractions (I)) will be called the *total mass* of the system of points.

Although we have defined the centre of mass with the aid of a coordinate system, we shall show that the position of the centre of mass does not depend on the coordinate system, but only on the masses of the points and their mutual distribution. This follows easily from the following theorem:

Theorem 1. *The statical moment of a system of points with respect to an arbitrary plane is equal to the statical moment of the total mass placed at the centre of gravity.*

Proof. Let Π be an arbitrary plane whose normal equation has the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$$

Let us take as positive that one of the two parts of space for which $x \cos \alpha + y \cos \beta + z \cos \gamma - p > 0$, when the coordinates of an arbitrary point of this part are substituted for x, y, and z. Hence, if $A(x, y, z)$ is an arbitrary point of space, then, since the distance of the point A from the plane Π is expressed by the formula

$$d = |x \cos \alpha + y \cos \beta + z \cos \gamma - p|,$$

according to our convention we can put

$$\sigma = x \cos \alpha + y \cos \beta + z \cos \gamma - p.$$

The statical moment of the point A of mass m with respect to the plane Π is therefore

$$M_\Pi = m\sigma = mx \cos \alpha + my \cos \beta + mz \cos \gamma - mp. \tag{1}$$

Let there be given a system of material points $m_1(x_1, y_1, z_1), \ldots, m_n(x_n, y_n, z_n)$. By (1) the statical moment of this system of points with respect to the plane Π will be

$$\begin{aligned} M_\Pi = {} & (m_1x_1 \cos \alpha + m_1y_1 \cos \beta + m_1z_1 \cos \gamma - m_1p) + \ldots \\ & \ldots + (m_nx_n \cos \alpha + m_ny_n \cos \beta + m_nz_n \cos \gamma - m_np); \end{aligned}$$

thus

$$\begin{aligned} M_\Pi = {} & (m_1x_1 + m_2x_2 + \ldots + m_nx_n) \cos \alpha + (m_1y_1 + \ldots + m_ny_n) \cos \beta + \\ & + (m_1z_1 + \ldots + m_nz_n) \cos \gamma - (m_1 + \ldots + m_n)\, p. \end{aligned} \tag{2}$$

Putting $m_1 + m_2 + \ldots + m_n = m$, we have by (I):

$$mx_0 = \sum m_ix_i, \quad my_0 = \sum m_iy_i, \quad mz_0 = \sum m_iz_i, \quad i = 1, 2, \ldots, n, \tag{II}$$

whence by (2)

$$M_\Pi = mx_0 \cos \alpha + my_0 \cos \beta + mz_0 \cos \gamma - mp. \tag{3}$$

Since the right hand side of equation (3) represents by (1) the statical moment of the mass m placed at the centre of gravity S having coordinates x_0, y_0, z_0, the theorem has been proved.

In order to show now that the centre of mass of the system of points does not depend on the choice of the coordinate system, let us suppose that another point S' possesses, in addition to point S, the property of the centre of gravity S described in the theorem. We shall prove that this is impossible.

With this end in view, let us pass through the point S an arbitrary plane Π not passing through S'. Therefore

$$M_{\Pi} = m\sigma \quad \text{and} \quad M_{\Pi} = m\sigma', \tag{4}$$

where σ and σ' denote the corresponding distances (positive or negative) of the points S and S' from the plane Π. From (4) it follows that $\sigma = \sigma'$. But $\sigma = 0$, since Π passes through S. Therefore σ' would also have to be equal to zero, which is impossible, because S' does not lie in the plane Π.

We see, therefore, that the *position of the centre of mass of a system of points is independent of the coordinate system.*

Centre of mass of two systems of points. Let a system U be composed of the material points

$$m_1'(x_1', y_1', z_1'), \quad m_2'(x_2', y_2', z_2',), \quad \ldots, \quad m_1''(x_1'', y_1'', z_1''), \quad m_2''(x_2'', y_2'', z_2''), \quad \ldots$$

The centre of mass S' of the system of points $m_1', m_2', \ldots$ has by definition the coordinates:

$$x_0' = (m_1'x_1' + m_2'x_2' + \ldots) / m', \quad y_0' = (m_1'y_1' + m_2'y_2' + \ldots) / m',$$
$$z_0' = (m_1'z_1' + m_2'z_2' + \ldots) / m', \tag{5}$$

where $m' = m_1' + m_2' + \ldots$ On the other hand, for the centre of mass S of the entire system U, there will be

$$x_0 = \frac{(m_1'x_1' + m_2'x_2' + \ldots) + (m_1''x_1'' + m_2''x_2'' + \ldots)}{(m_1' + m_2' + \ldots) + (m_1'' + m_2'' + \ldots)},$$

whence by (5)

$$x_0 = \frac{m'x_0' + (m_1''x_1'' + m_2''x_2'' + \ldots)}{m' + (m_1'' + m_2'' + \ldots)}. \tag{6}$$

Similar formulae are obtained for y_0 and z_0. Formula (6) represents the x-coordinate of the centre of mass of the system that is obtained from the given system U, if a part of it, namely the points of masses $m_1', m_2', \ldots$, is replaced by a single material point of mass $m' = m_1' + m_2' + \ldots$ placed at the centre of mass of this part. Therefore, we have obtained

Theorem 2. *The centre of mass of a system of points is not altered if a part of it is replaced by a material point having a mass equal to the mass of this part and placed at centre of its mass.*

Hence if we have in particular two systems of points U' and U'' of total masses m' and m'' and with centres of gravity S' and S'', then we obtain the centre of mass of the system $U' + U''$ by determining the centre of mass of the system of two material points having masses m' and m'' placed at the points S' and S'', respectively. This is so because the systems U' and U'' can be considered as parts of the system $U' + U''$.

Plane system of points. A system of material points is said to be a *plane* system if all points of the system lie in one plane. Selecting this plane as the xy-plane (which is possible for us to do since the centre of mass is independent of the choice of the coordinate system), we shall have for the points of the system $z_1 = 0$, $z_2 = 0$, $\ldots$, $z_n = 0$, whence by formulae (I), p. 152, we get $z_0 = 0$.

The centre of gravity of a plane system therefore lies in the plane of the system.

The statical moment of a plane system with respect to an arbitrary line l lying in the plane of the system is defined by the expression

$$M_l = \sum_{i=1}^{n} m_i \sigma_i, \tag{7}$$

where $|\sigma_i|$ is the distance of the point of mass m_i from l, and the sign of σ_i depends on whether m_i is situated in the positive or negative parts of the plane into which this plane is divided by the line l. We see from this that the statical moment of a plane system with respect to the line l is the statical moment of this system with respect to a plane perpendicular to the plane of the system and intersecting it along the line l. Hence, in particular, the moments with respect to the x and y axes are expressed by the formulae:

$$M_x = \Sigma m_i y_i, \quad M_y = \Sigma m_i x_i. \tag{8}$$

Linear system of points. If the points of a system lie on one line l, then the centre of mass of the system also lies on this line, because choosing the line l as the x-axis, we have $y_1 = 0$, $y_2 = 0$, $\ldots$ and $z_1 = 0$, $z_2 = 0, \ldots$, whence by formulae (I), p. 152, we get $y_0 = 0$, $z_0 = 0$. The centre of mass will therefore also lie on the x-axis.

Centre of mass of two points. Let the material points of masses m_1 and m_2 be at a distance d from each other. The centre of mass obviously lies on a line joining these points. Let us place at m_1 the origin of the x-axis, and pass its positive part through m_2. The points m_1 and m_2 will therefore have the coordinates $x_1 = 0$ and $x_2 = d$, respectively, and the centre of mass the coordinate

$$x_0 = m_2 d \,/\, (m_1 + m_2). \tag{9}$$

Since $0 < x_0 < d$, the centre of mass lies between the points. Denoting the distances of the centre of mass from the points m_1 and m_2 by d_1 and d_2, respectively, we obtain $d_1 = x_0 = m_2 d \,/\, (m_1 + m_2)$ and $d_2 = d - d_1 = m_1 d \,/\, (m_1 + m_2)$, whence

$$d_1 : d_2 = m_2 : m_1. \tag{10}$$

Hence we have the following

Theorem 3. *The centre of mass of a two point system lies between the points of the system and divides the line segment joining these points in the inverse ratio to their masses.*

Making use of theorem 2, p. 154, we can determine the centre of mass of a finite system of points $A_1, A_2, A_3, \ldots$ of masses $m_1, m_2, m_3, \ldots$ (Fig. 106) in the following manner: we determine at first the centre of mass S_1 of the system of points A_1 and A_2; next we determine the centre of mass S_2 of the system of two points consisting of the point of mass $m_1 + m_2$, situated at S_1 and of the point A_3 of mass m_3; continuing in this manner, we obtain the centre of mass of the entire given system.

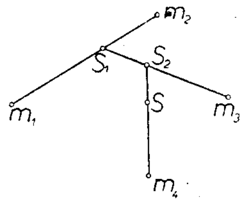

Fig. 106.

Symmetric systems of points. A point O (line l, plane Π) is called a *centre* (*line*, *plane*) *of symmetry of the system of material points*, if to each point m_i there exists in the system a point having the same mass m_i, placed symmetrically with respect to point O (line l, plane Π).

If the centre of symmetry is the origin of the coordinate system (Fig. 107), then along with each point $m_i(x_i, y_i, z_i)$ the system of points includes the point $m_i(-x_i, -y_i, -z_i)$. If the plane of symmetry is the xy-plane (Fig. 108), then along with each point $m_i(x_i, y_i, z_i)$ the system

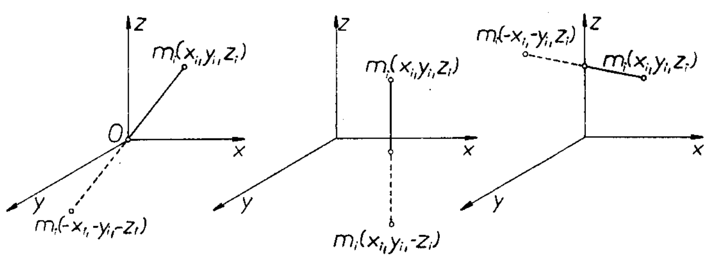

Fig. 107. Fig. 108. Fig. 109.

includes the point $m_i(x_i, y_i, -z_i)$. If the axis of symmetry is the z-axis (Fig. 109), then along with each point $m_i(x_i, y_i, z_i)$ the system includes the point $m_i(-x_i, -y_i, z_i)$.

It is easy to show that the *centre of symmetry is always the centre of mass.*

For by theorem 3, the centre of mass of a pair of symmetric points lies at the centre of symmetry. Hence by theorem 2, p. 154, we can replace such a pair by a material point situated at the centre of symmetry. Doing this with every pair, we come to the conclusion that the centre of symmetry is the centre of mass of the entire system.

Similarly, the *centre of mass lies on a line of symmetry* (*and on a plane of symmetry*).

Because for these same reasons the centre of a pair of symmetric points lies on a line (and on a plane) of symmetry.

§ 3. Moments of second order. Moment of inertia. Let there be given a material point A of mass m and a certain plane Π. Let r denote the distance of the point A from the plane Π. The expression

$$I = mr^2 \tag{1}$$

is called the *moment of inertia* of the point A with respect to the plane Π. If we denote by r the distance of the material point A from a certain line l (or from a certain point O), then (1) will be the moment of inertia of the point A with respect to the line l (or with respect to the point O).

The total moment of inertia of a system of points is defined as the sum of the moments of inertia of the separate points of this system.

Product of inertia. Let there be given two mutually perpendicular planes Π_1 and Π_2. Put $\sigma_1 = \pm d_1$, where d_1 denotes the distance of the material point A from Π_1, and the sign depends on whether the point is in the positive or negative of the two parts into which the plane Π_1 divides space. We define σ_2 with respect to the plane Π_2 similarly. The expression

$$D = m\sigma_1\sigma_2 \tag{2}$$

is called the *product of inertia* of the material point A with respect to the planes Π_1 and Π_2.

The *total product of inertia* of a system of points $A_1, A_2, \ldots$ with respect to the planes Π_1 and Π_2 is defined as the sum of the products of inertia of the separate points. Hence

$$D = \Sigma m_i \sigma_1^{(i)} \sigma_2^{(i)}, \tag{3}$$

where m_i, $\sigma_1^{(i)}$, $\sigma_2^{(i)}$ denote respectively the mass of the point A_i and its distances from the planes Π_1 and Π_2 (preceded by proper signs).

Moments of inertia and products of inertia are called *moments of second order*.

Radius of gyration. Let I denote the total moment of inertia of a system of points U with respect to a plane Π (line l, point O). The number

$$k = \sqrt{I / m}, \quad \text{where } m = m_1 + m_2 + \ldots \tag{4}$$

is called the *radius of gyration* of the system of points U with respect to the plane Π (line l, point O). In virtue of (4)

$$I = mk^2. \tag{5}$$

Therefore: *the radius of gyration k is the distance at which the total mass of a system has a moment of inertia equal to the total moment of inertia of the system.*

Concentrated mass. Let r be an arbitrary positive number. The mass of a system *concentrated at a distance* r with respect to a plane (line or point) is defined by the number

$$m_r = I / r^2. \tag{6}$$

Therefore $I = m_r r^2$. Hence: *the moment of inertia of a system with respect to a plane (line, point) is equal to the moment of inertia of its concentrated mass m_r situated at a distance r from this plane (line, point).*

The moments of second order of the system of points

$$m_1(x_1, y_1, z_1), \quad m_2(x_2, y_2, z_2), \quad \ldots, \quad m_n(x_n, y_n, z_n)$$

with respect to a plane, axis and the origin of a coordinate system are expressed by means of the following formulae:

The moments of inertia with respect to the planes xy, yz and xz:

$$I_{xy} = \sum_{i=1}^{n} m_i z_i^2, \quad I_{yz} = \sum_{i=1}^{n} m_i x_i^2, \quad I_{zx} = \sum_{i=1}^{n} m_i y_i^2. \tag{7}$$

The moments of inertia with respect to the x, y, and z axes:

$$I_x = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2), \; I_y = \sum_{i=1}^{n} m_i(x_i^2 + z_i^2), \; I_z = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2). \tag{8}$$

The moments of inertia with respect to the origin O of a coordinate system:

$$I_o = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2 + z_i^2). \tag{9}$$

The products of inertia with respect to the pairs of planes xy and zx, xy and yz, as well as xz and zy:

$$D_x = \sum_{i=1}^{n} m_i y_i z_i, \; D_y = \sum_{i=1}^{n} m_i x_i z_i, \; D_z = \sum_{i=1}^{n} m_i x_i y_i. \tag{10}$$

From formulae (7)—(10) the following relations can be easily derived:

$$I_x = I_{xy} + I_{xz}, \quad I_y = I_{xy} + I_{yz}, \quad I_z = I_{xz} + I_{yz};$$
$$I_{xy} = \tfrac{1}{2}[I_x + I_y - I_z], \quad I_{yz} = \tfrac{1}{2}[I_y + I_z - I_x], \quad I_{xz} = \tfrac{1}{2}[I_x + I_z - I_y];$$
$$I_o = \tfrac{1}{2}[I_x + I_y + I_z] = I_{xy} + I_{yz} + I_{zx},$$
$$I_o = I_x + I_{yz} = I_y + I_{xz} = I_z + I_{xy}.$$

Moments of inertia with respect to parallel lines. Knowing the moments of inertia of a system of points of total mass m with respect to lines passing through one point e. g. through the origin of the coordinate system, we can easily determine the moment of inertia of this system of points with respect to an arbitrary line in space by making use of the following theorem:

If a line l passing through the centre of mass of a system of points is parallel to the line l', then

$$I_{l'} = I_l + md^2, \tag{I}$$

where d denotes the distance between l and l', whereas I_l and $I_{l'}$ denote the moments of inertia with respect to these lines.

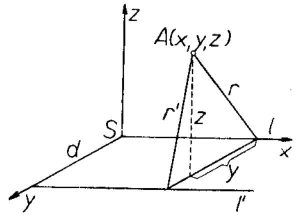

Fig. 110.

Proof. Let us choose the centre S of the mass of the system of points through which the line l passes as the origin of the coordinate system, the line l as the x-axis, and the plane passed through the parallel lines l and l' as the xy-plane (Fig. 110). Denoting by r and r' respectively the distances of an arbitrary point $A(x, y, z)$ from the straight lines l and l', we have $r'^2 = z^2 + (d - y)^2$ and $r^2 = z^2 + y^2$, whence $r'^2 = r^2 + d^2 - 2dy$, and hence

$$I_{l'} = \sum_{i=1}^{n} m_i r_i'^2 = \sum_{i=1}^{n} m_i [r_i^2 + d^2 - 2dy_i] =$$
$$= \sum_{i=1}^{n} m_i r_i^2 + d^2 \sum_{i=1}^{n} m_i - 2d \sum_{i=1}^{n} m_i y_i = I_l + md^2 - 2d \sum_{i=1}^{n} m_i y_i.$$

But $\sum_{i=1}^{n} m_i y_i = my_0 = 0$, since the centre S of the mass of the system of points lies at the origin of the system of coordinates. Therefore $I_{l'} = I_l + md^2$, q. e. d.

From formula (I) it follows that if all lines parallel to a line having a certain given direction are taken into consideration, then the moment of inertia will be the least with respect to that line which passes through the centre of mass. It is equally obvious that if lines l' and l'' are parallel, then denoting by d_1 and d_2 the distances of the centre of mass from these lines, we shall have

$$I_{l'} - md_1^2 = I_{l''} - md_2^2, \tag{11}$$

because by (I) both sides of the equality are equal to I_l, where l is a line parallel to l' and l'' and passing through the centre of mass.

From formula (11) we have

$$I_{l''} = I_{l'} + m(d_2^2 - d_1^2). \tag{II}$$

This formula enables one to compute the moment of inertia of a system of points with respect to an arbitrary line in space, if the moments of inertia of this system with respect to every line passing through one point and the position of the centre of mass of the system are known.

Products of inertia with respect to parallel planes. For products of inertia we can prove a theorem similar to the theorem on moments of inertia (p. 159).

Let the planes Π_1, Π_2 be perpendicular to each other and pass through the centre of mass m of a given system of material points. Let us select arbitrary planes Π_1', Π_2' parallel to the planes Π_1, Π_2, respectively. Let σ_1 denote the distance between the planes Π_1, Π_1', preceded by a $+$ or $-$ sign depending on whether the plane Π_1' lies in the positive or negative part of space into which the plane Π_1 divides space. Let us define σ_2 for the pair of planes Π_2, Π_2' analogously. Finally, let us denote the products of inertia of the given system with respect to the pairs of planes Π_1, Π_2 and Π_1', Π_2' by D and D'. We then have a formula which is analogous to (I), namely:

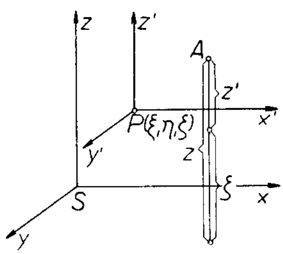

Fig. 111.

$$D' = D + m\sigma_1\sigma_2. \tag{III}$$

Remark. Let us note that the product $m\sigma_1\sigma_2$ denotes the product of inertia with respect to the pair of planes Π_1, Π_2 of the total mass m of the system placed anywhere on the intersection of the pair of planes Π_1' and Π_2'.

Proof of formula (III). Let us choose the origin of the coordinate system (x, y, z) at the centre S of the total mass m of the given system of points (Fig. 111). As the x-axis we shall take the intersection of the planes Π_1 and Π_2, and we select these planes as the xy and xz planes, respectively.

Analogously, we select a second system of coordinates (x', y', z') for the pair of planes Π_1', Π_2', taking as the origin an arbitrary point P lying on the line of intersection of the planes Π_1' and Π_2'.

Let us denote the coordinates of the point P with respect to the coordinate system (x, y, z) by ξ, η, ζ. Obviously $\eta = \sigma_2$, and $\zeta = \sigma_1$. Let x, y, z be the coordinates of an arbitrary point A with respect to the coordinate system (x, y, z), and x', y', z' the coordinates of this point A with respect to the coordinate system (x', y', z'). Then:

$$x' = x - \xi,\ y' = y - \eta,\ z' = z - \zeta. \tag{12}$$

Since

$$D' = \Sigma m_i z'_i y'_i,\ D = \Sigma m_i z_i y_i, \tag{13}$$

it follows by (11) that

$$D' = \Sigma m_i (z_i - \zeta)(y_i - \eta) = \Sigma m_i [z_i y_i + \zeta\eta - \zeta y_i - \eta z_i] = \tag{14}$$
$$= \Sigma m_i z_i y_i + \zeta\eta\, \Sigma m_i - \zeta\, \Sigma m_i y_i - \eta\, \Sigma m_i z_i.$$

But $\Sigma m_i y_i = m y_0 = 0$, and $\Sigma m_i z_i = m z_0 = 0$, because by hypothesis the centre S of mass m of the given system of points lies at the origin of the system (x, y, z). Therefore, by (13) and (14), $D' = D + m\zeta\eta$, and since $\zeta = \sigma_1$, and $\eta = \sigma_2$, we obtain finally $D' = D + m\sigma_1\sigma_2$, q. e. d.

§ 4. Ellipsoid of inertia. Principal axes of inertia. Let $O(x, y, z)$ be an arbitrary rectangular coordinate system with origin at O. We shall prove that it is possible to determine the moments of inertia with respect to an arbitrary line l passing through O, if the moments of inertia with respect to the axes and the products of inertia with respect to the planes of this coordinate system are known.

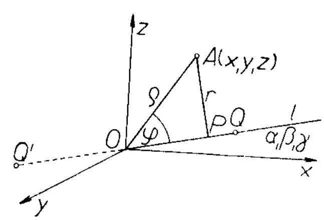

Fig 112.

Let the line l form the angles α, β, γ with the axes of the coordinate system $O(x, y, z)$ (Fig. 112). Let us select an arbitrary point $A(x, y, z)$ and let P be the projection of the point A on the line l. Therefore $AP = r$ is the distance of the point A from the line l. Let us put

$$OA = \varrho = \sqrt{x^2 + y^2 + z^2}. \tag{1}$$

Denoting by φ the angle between the line OA and the line l, we obtain

$$AP = r = \varrho \sin\varphi. \tag{2}$$

Since

$$OP = x\cos\alpha + y\cos\beta + z\cos\gamma,$$

as is known from analytic geometry, and because $OP = \varrho\cos\varphi$, we obtain

$$\cos\varphi = (x\cos\alpha + y\cos\beta + z\cos\gamma)\,/\,\varrho.$$

By (2) $r^2 = \varrho^2\sin^2\varphi = \varrho^2(1 - \cos^2\varphi)$; hence

$$r^2 = \varrho^2\left[1 - \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)^2}{\varrho^2}\right] =$$
$$= \varrho^2 - [x\cos\alpha + y\cos\beta + z\cos\gamma]^2,$$

whence by (1),

$$r^2 = x^2[1 - \cos^2\alpha] + y^2[1 - \cos^2\beta] + z^2[1 - \cos^2\gamma] -$$
$$- 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma.$$

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, substituting $1 - \cos^2\alpha = \cos^2\beta + \cos^2\gamma$, etc. we obtain

$$r^2 = (y^2 + z^2)\cos^2\alpha + (x^2 + z^2)\cos^2\beta + (x^2 + y^2)\cos^2\gamma -$$
$$- 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma.$$

Denoting by m_i the mass and by r_i the distance of the point A_i of a given system of points $A_1, A_2, \ldots$ from the line l, we obtain for the moment of inertia I_l of this system of points with respect to the line l the formula

$$I_l = \Sigma m_i r_i^2 =$$
$$= \cos^2\alpha\,\Sigma m_i(y_i^2 + z_i^2) + \cos^2\beta\,\Sigma m_i(x_i^2 + z_i^2) + \cos^2\gamma\,\Sigma m_i(x_i^2 + y_i^2) -$$
$$- 2\cos\alpha\cos\beta\,\Sigma m_i x_i y_i - 2\cos\alpha\cos\gamma\,\Sigma m_i x_i z_i - 2\cos\beta\cos\gamma\cdot\Sigma m_i y_i z_i,$$

whence

$$I_l = I_x\cos^2\alpha + I_y\cos^2\beta + I_z\cos^2\gamma - \tag{I}$$
$$- 2D_z\cos\alpha\cos\beta - 2D_y\cos\alpha\cos\gamma - 2D_x\cos\beta\cos\gamma.$$

From formula (I) we can determine the moment of inertia of a system of material points with respect to the line l, if we know $I_x, I_y, I_z, D_x, D_y, D_z$ as well as the angles which the line l forms with the axes of the coordinate system.

Retaining the previous notation, let us denote the length of the radius of gyration of a given system of points with respect to l by k_l. Therefore (p. 157)

$$k_l = \sqrt{I_l / m}. \tag{3}$$

Let us assume that $I_l \neq 0$ for every line l passing through O. This assumption is equivalent to the assumption that the material points of the given system do not lie on one and the same line passing through O. Since $I_l \neq 0$, it follows by (3) that also $k_l \neq 0$.

On each line l, let us cut off segments OQ and OQ' (Fig. 113) whose lengths are inversely proportional to the radius of gyration k_l, i. e.

$$OQ = OQ' = a / k_l = a\sqrt{m / I_l}, \tag{4}$$

where a is an arbitrary positive constant independent of the line l. Denoting the coordinates of the point Q by x, y, z, we have:

$$x = \pm a\sqrt{m / I_l}\cos\alpha, \quad y = \pm a\sqrt{m / I_l}\cos\beta, \\ z = \pm a\sqrt{m / I_l}\cos\gamma. \tag{5}$$

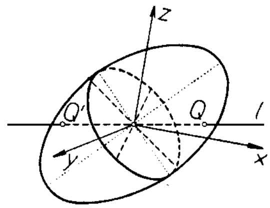

Fig. 113.

The point Q' has the coordinates $-x, -y, -z$. The collection of all points Q and Q' will form a certain surface Σ. In order to obtain its equation we shall determine $\cos\alpha$, $\cos\beta$, $\cos\gamma$ from (5) and substitute the values obtained into equation (I). We get

$$I_l = \frac{I_l}{ma^2}[I_x x^2 + I_y y^2 + I_z z^2 - 2D_x yz - 2D_y xz - 2D_z xy],$$

whence

$$I_x x^2 + I_y y^2 + I_z z^2 - 2D_x yz - 2D_y xz - 2D_z xy = c^2, \tag{6}$$

where $c^2 = ma^2$. We see from this that the surface Σ is of the second degree.

We shall show that Σ is an ellipsoid. For this purpose it is sufficient to prove that Σ is a bounded surface (i. e. that the distances of its points from the origin of the coordinate system do not exceed a certain number). Indeed, we have assumed that $I_l \neq 0$, and hence we have $I_l > 0$ constantly. Since by formula (I) I_l is a continuous function of the angles α, β, γ, the minimum of I_l is also positive. Denoting this minimum by h, we have by (3) $k_l \geq \sqrt{h / m}$, whence by (4), $OQ' = OQ \leq a\sqrt{m / h}$. Therefore the surface Σ is a bounded surface. It follows from this that the surface Σ is an ellipsoid, because the only bounded surface of the second degree is an ellipsoid.

The ellipsoid Σ is called the *ellipsoid of inertia* of the given system of points with respect to the point O.

Since equation (6) lacks terms of the first degree, i. e. x, y, z, the point O is the centre of the ellipsoid of inertia.

Therefore: *the ellipsoid of inertia of a system of material points with respect to a point O has this property, that the distance of each of its points from O is inversely proportional to the radius of gyration of the system with respect to the diameter passing through this point.*

The axes of the ellipsoid of inertia with respect to the point O are called the *principal axes of inertia* with respect to the point O.

The ellipsoid of inertia with respect to the centre of gravity is called the *central* ellipsoid of inertia, its axes, *central* axes of inertia, and the planes passing through two axes, *central planes.*

Obviously, there exist infinitely many ellipsoids of inertia of a given system with respect to one and the same point O. They depend on the choice of the constant of proportionality. However, all these ellipsoids nevertheless have a common centre and common directions of the principal axes. In addition, the ratio of the principal axes is the same for all ellipsoids. All the ellipsoids of inertia with respect to one and the same point O are therefore similar to each other.

The radius of gyration has the greatest value with respect to the minor axis; hence the moment of inertia has the smallest value with respect to the minor axis. The converse is true with respect to the major axis.

In particular, if the ellipsoid is a sphere, the point O is termed a *spherical* point.

The moments of inertia with respect to every line passing through a spherical point are the same (and conversely).

Determination of principal axes of inertia. If we take the principal axes of inertia as the coordinate axes x, y, z, then the equation of the ellipsoid of inertia will have the form

$$Ax^2 + By^2 + Cz^2 = F. \tag{7}$$

Comparing equations (6) and (7), we see that in this case $D_x = 0$, $D_y = 0$ and $D_z = 0$; therefore the ellipsoid of inertia will be

$$I_x x^2 + I_y y^2 + I_z z^2 = c^2. \tag{8}$$

Hence: *the necessary and sufficient condition that the coordinate axes be axes of inertia is that the products of inertia D_x, D_y, D_z be equal to zero.*

If the coordinate axes x, y, z are selected so that only one of them, e. g. the z-axis, coincides with one of the principal axes of inertia, then the equation of the ellipsoid will have the form

$$Ax^2 + By^2 + Cz^2 + Exy = F. \tag{9}$$

Comparing equations (6) and (9), we see that $D_x = 0$ and $D_y = 0$. The equation of the ellipsoid of inertia in this case will therefore be

$$I_x x^2 + I_y y^2 + I_z z^2 - 2D_z xy = c^2. \tag{10}$$

Hence: *the necessary and sufficient condition that the z-axis be a principal axis of inertia is that the products of inertia D_x and D_y be equal to zero.*

It is easy to formulate analogous conditions that the principal axes of inertia be the x-axis or the y-axis.

Let us now take the centre S of the total mass m of a given system of material points as the origin of the coordinate system (x, y, z), and its central axes as the coordinate axes x, y, z. We obviously have

$$D_x = 0, \quad D_y = 0, \quad D_z = 0. \tag{11}$$

Let us next take an arbitrary point $O(\xi, \eta, \zeta)$ as the origin of a new coordinate system (x', y', z') whose axes are parallel to the axes x, y and z, respectively. By (III), p. 160,

$$D_{x'} = D_x + m\zeta\eta, \quad D_{y'} = D_y + m\xi\zeta, \quad D_{z'} = D_z + m\xi\eta,$$

whence by (11):

$$D_{x'} = m\zeta\eta, \quad D_{y'} = m\xi\zeta, \quad D_{z'} = m\xi\eta. \tag{12}$$

Let us assume that the point $O(\xi, \eta, \zeta)$ lies on one of its central planes, e. g. on the xy-plane. Therefore $\zeta = 0$. Hence by (12) we obtain $D_{x'} = 0$ and $D_{y'} = 0$. It follows from this that the z'-axis is a principal axis of inertia with respect to the point O. Hence we have the theorem:

One of the principal axes of inertia with respect to a point lying in a central plane is perpendicular to this plane.

In particular, if the point O lies on a central axis, e. g. on the x-axis, then $\eta = 0$ and $\zeta = 0$, whence by (12) $D_{x'} = 0$, $D_{y'} = 0$ and $D_{z'} = 0$. It follows from this that the x', y', z' axes are principal axes of inertia with respect to the point O. We therefore obtain the corollaries:

1° *The principal axes of inertia with respect to a point lying on a central axis are parallel to the central axes.*

2° *The central axis is the principal axis of inertia with respect to each of its points.*

If the given system of material points possesses an axis or a plane of symmetry, then we can prove the following theorem:

An axis of symmetry of a system of material points is a central axis; similarly, *a plane of symmetry is a central plane.*

Proof. In order to prove the first part of the theorem, let us note that the centre S of mass m of the system lies on the axis of symmetry. Let us take S as the origin of the coordinate system (x, y, z) and the axis of symmetry as the z-axis. Because of this the given system of material points includes in addition to each point A_i of mass m_i and coordinates x_i, y_i, z_i, another point A_i' of mass equal to m_i and having coordinates $-x_i, -y_i, z_i$. We therefore have

$$D_x = \Sigma[m_i y_i z_i + m_i(-y_i)\, z_i] = 0, \quad D_y = \Sigma[m_i x_i z_i + m_i(-x_i)\, z_i] = 0.$$

It follows from this that the axis of symmetry z is at the same time the principal axis of inertia with respect to the centre of mass S, and therefore it is a central axis.

In order to prove the second part of the theorem, let us note that the centre of mass S lies in the plane of symmetry. Let us take the origin of the coordinate system at S and the x and y axes in the plane of symmetry. Since the xy-plane is a plane of symmetry, to each point A_i of mass m_i and coordinates x_i, y_i, z_i, there exists in our system of material points a point A_i' of mass equal to m_i and having coordinates $x_i, y_i, -z_i$. We therefore have

$$D_x = \Sigma[m_i y_i z_i + m_i y_i(-z_i)] = 0, \; D_y = \Sigma[m_i x_i z_i + m_i x_i(-z_i)] = 0.$$

It follows from this that the z-axis is a central axis, and hence the plane of symmetry xy, being perpendicular to the central axis z, is a central plane.

§ 5. Second moments of a plane system. Let a system of material points lying in a plane Π be given. Since the plane Π is a plane of symmetry, by the preceding theorem it is a central plane. Therefore, at every point of the plane Π, one of the principal axes of inertia is perpendicular to the plane Π, while the two remaining principal axes of inertia lie in the plane Π.

If only the moments of inertia with respect to lines lying in the plane Π are taken into account, then the considerations of §§ 3 and 4 can be simplified.

Let us choose the given point O as the origin of the coordinate system (x, y) of the plane Π. From the point O let us draw an arbitrary line l lying in this plane and forming an angle α with the x-axis (Fig. 114). The moment of inertia with respect to l is obtained from formula (I), p. 162, by putting $\beta = 90° - \alpha$, and $\gamma = 90°$. Consequently

$$I_l = I_x \cos^2\alpha + I_y \sin^2\alpha - D_z \sin 2\alpha. \tag{1}$$

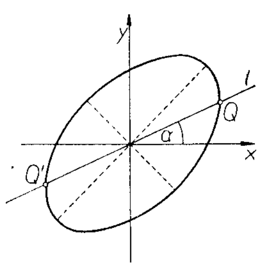

Fig. 114.

We shall call the product of inertia $D_z = \Sigma m_i x_i y_i$ the *product of inertia with respect to the x and y axes.*

On the line l let us mark off the points Q and Q' whose distances from O are inversely proportional to the radius of gyration. The collection of such pairs of points marked off on all lines l which pass through O will form a curve which will be the intersection of the plane Π with the ellipsoid of inertia with respect to the point O. This curve will therefore be an ellipse; we shall call it the *ellipse of inertia with respect to O.*

Its equation is obtained from equation (6), p. 163, by putting $z = 0$. Hence the equation of the ellipse of inertia will be

$$I_x x^2 + I_y y^2 - 2D_z xy = c^2. \tag{2}$$

If the x and y axes are chosen as the principal axes of inertia, then the equation of the ellipse will assume the form

$$I_x x^2 + I_y y^2 = c^2. \tag{3}$$

II. SOLIDS, SURFACES AND MATERIAL LINES

§ 6. Density. If a body is not so small that it can be considered as a material point, then, in addition to the mass of the body, we also give the distribution of the mass in the body; for in many problems of mechanics, not only a knowledge of the mass of the entire body is of great importance, but also a knowledge of the mass of its separate parts.

Frequently it happens that the mass of a part of a body is proportional to the volume. Then, denoting the mass by m, and the volume of the body by v, we obtain as the mass per unit volume

$$\varrho = m / v. \tag{I}$$

The number ϱ is called the *density* of the body.

In this case we say that the mass of the body is distributed uniformly, or that the body is *homogeneous*, or finally, that the *density is constant.*

The mass of a part of a body of volume v' is then $m' = v'\varrho$. By (I) the dimension of density is

$$[\text{density}] = L^{-3}M. \tag{1}$$

Let us pass on now to the general case, i. e. we do not assume that the mass in a given body is distributed uniformly. Let A be any point of the given body. Let us select in the body an arbitrary cube of mass Δm and volume Δv, whose centre is the point A. The limit

$$\lim_{\Delta v \to 0} \frac{\Delta m}{\Delta v} = \varrho \tag{II}$$

is called the *density of the body at the point A*.

In general, the density ϱ depends on the point A. If A has the coordinates x, y, z, then ϱ is a function of the variables x, y, z. We can therefore write $\varrho = \varrho(x, y, z)$. If $\varrho = \text{const}$, then the mass of the body is distributed uniformly, and we have the case of the homogeneous body already considered.

We shall always assume that ϱ is a continuous function.

Calculation of mass. Knowing the density at every point of a body, we can calculate its mass as well as the mass of an arbitrary part of it.

By means of planes parallel to the xy, yz and zx planes, let us divide the given body into small rectangular parallelepipeds (the so-called *elements of volume*) and possibly into boundary pieces of irregular form. Let us denote the volumes of successive rectangular parallelepipeds by $\Delta v_1, \Delta v_2, \ldots$ and let us select one point in each one of them whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ respectively. The masses of the separate rectangular parallelepipeds are approximately $\varrho(x_1, y_1, z_1)\, \Delta v_1$, $\varrho(x_2, y_2, z_2)\, \Delta v_2, \ldots$ Therefore the sum

$$\varrho(x_1, y_1, z_1)\, \Delta v_1 + \varrho(x_2, y_2, z_2)\, \Delta v_2 + \ldots$$

represents approximately the mass of the body. Forming subdivisions into smaller and smaller rectangular parallelepipeds whose dimension tend to zero and passing to the limit, we obtain for the mass m of a given body the formula

$$m = \iiint_D \varrho(x, y, z)\, \mathrm{d}v, \tag{III}$$

where the region of integration D extends over the entire body.

In particular, if $\varrho = \text{const}$, we obtain from formula (III) $m = \varrho v$ which agrees with formula (I).

Formula (III) also gives the mass of an arbitrary part of the given body if we assume that D denotes the region occupied by this part.

Material surface, material line. Sometimes one or two dimensions of a body are small in comparison with the remaining ones. Examples of such bodies are plates, rods, wires etc. In these cases we represent the body as a surface or a material line and say that its *mass* is *distributed along a surface* or *along a line*.

Let a mass be distributed along a surface S, and let A denote an arbitrary point on this surface. Let us denote by $\Delta\sigma$ the area of a small part of the surface S containing the point A (the so-called *element of area*), and by Δm the mass of this part. If the dimensions of the element tend to zero, then

$$\lim_{\Delta\sigma\to 0}\frac{\Delta m}{\Delta\sigma}=\varrho \tag{IV}$$

is called the *surface density at the point A*.

In particular, if $\varrho = \text{const}$, then ϱ represents the mass of an element 1 cm² in area cut out from the surface S.

It can be shown (in a manner similar to that used for solids) that the mass of the surface S is expressed by the formula

$$m = \iint_S \varrho \, d\sigma, \tag{V}$$

where the region of integration S extends over the entire surface.

If $\varrho = \text{const}$, and P denotes the area of the surface S, we have $m = \varrho P$, whence

$$\varrho = m / P. \tag{2}$$

From the above formula we obtain as the dimension of surface density

$$[\text{surface density}] = L^{-2}M. \tag{3}$$

We proceed similarly in the case of a mass distributed linearly along a certain curve C. If A is a point of the curve C, then we choose an arbitrary arc of the curve C containing the point A. If Δs denotes the length of this arc (the so-called *element of length*), and Δm its mass, then

$$\lim_{\Delta s\to 0}\frac{\Delta m}{\Delta s}=\varrho \tag{VI}$$

is called the *linear density at the point A*.

In particular, if $\varrho = \text{const}$, then ϱ represents the mass of an arc (of the curve C) 1 cm in length.

The mass of the entire curve C is

$$m = \int_C \varrho \, ds, \tag{VII}$$

where the region of integration extends over the entire curve.

If $\varrho = \text{const}$, and s denotes the length of the curve C, we have by (VII):

$$m = \varrho s, \quad \text{whence} \quad \varrho = m / s. \tag{4}$$

Hence, as the dimension of linear density, we obtain the formula

$$[\text{linear density}] = L^{-1}M. \tag{5}$$

§ 7. Statical moments and moments of inertia. Centre of mass. The *statical moment* of a body with respect to a certain plane, e. g. the xy-plane, is defined in the following manner. We divide the body into small rectangular parallelepipeds of volumes Δv_1, Δv_2, ... and possibly into certain irregular boundary pieces. In each rectangular parallelepiped we select arbitrarily one point whose coordinates are $x_1, y_1, z_1,\ x_2, y_2, z_2, \ldots$ respectively. Let us denote the density of the body at the point x, y, z, by $\varrho(x, y, z)$. The mass of the rectangular parallelepiped Δv_1 is approximately $\varrho(x_1, y_1, z_1) \cdot \Delta v_1$; if its entire mass were situated at the point x_1, y_1, z_1, then the statical moment of this mass with respect to the xy-plane would be equal to $z_1 \cdot \varrho(x_1, y_1, z_1) \cdot \Delta v_1$. We can therefore consider the sum

$$z_1 \cdot \varrho(x_1, y_1, z_1) \cdot \Delta v_1 + z_2 \cdot \varrho(x_2, y_2, z_2) \cdot \Delta v_2 + \ldots$$

as representing approximately the statical moment of the body with respect to the xy-plane. It is for this reason that the limit of the above sum, when the dimensions of the rectangular parallelepipeds tend to zero, is called the *statical moment of the body with respect to the xy-plane.*

Since the limit of the above sum is the triple integral

$$\iiint_D z\varrho \,\mathrm{d}v,$$

where the region of integration D extends over the entire body, we have

$$M_{xy} = \iiint_D z\varrho \,\mathrm{d}v, \text{ and similarly } M_{xz} = \iiint_D y\varrho \,\mathrm{d}v,\ M_{yz} = \iiint_D x\varrho \,\mathrm{d}v. \quad \text{(I)}$$

We define the statical moment of surfaces and curves analogously. Instead of triple integrals there occur double integrals over surfaces and single integrals over curves.

In the case of a mass distributed over a surface S we obtain

$$M_{xy} = \iint_S z\varrho \,\mathrm{d}\sigma, \quad M_{xz} = \iint_S y\varrho \,\mathrm{d}\sigma, \quad M_{yz} = \iint_S x\varrho \,\mathrm{d}\sigma, \qquad \text{(II)}$$

where $\mathrm{d}\sigma$ is an element of area, and for a mass distributed linearly along a curve C:

$$M_{xy} = \int_C z\varrho \,\mathrm{d}s, \quad M_{xz} = \int_C y\varrho \,\mathrm{d}s, \quad M_{yz} = \int_C x\varrho \,\mathrm{d}s, \qquad \text{(III)}$$

where $\mathrm{d}s$ is an element of arc length.

The statical moments of plane figures with respect to the x and y axes are expressed by the formulae

$$M_x = \iint_D y\varrho \,\mathrm{d}x\,\mathrm{d}y, \quad M_y = \iint_D x\varrho \,\mathrm{d}x\,\mathrm{d}y. \qquad (1)$$

For plane curves we have:

$$M_x = \int_C y\varrho \,\mathrm{d}s, \quad M_y = \int_S x\varrho \,\mathrm{d}s. \qquad (2)$$

The *centre of mass* of a body, surface or a material curve is defined as the point having coordinates:

$$x_0 = M_{zy} / m, \quad y_0 = M_{xz} / m, \quad z_0 = M_{xy} / m, \tag{IV}$$

where M_{zy}, M_{xz}, M_{xy} denote the statical moments with respect to the zy, xz, xy planes, and m is the mass of the body.

For plane figures and curves we get:

$$x_0 = M_y / m, \quad y_0 = M_x / m. \tag{V}$$

If the density of the body $\varrho =$ const, then

$$M_{zy} = \varrho \iiint\limits_D x \, \mathrm{d}v, \quad \text{and} \quad m = \varrho \iiint\limits_D \mathrm{d}v = \varrho v,$$

whence

$$x_0 = \frac{\iiint\limits_D x \, \mathrm{d}v}{v}, \quad y_0 = \frac{\iiint\limits_D y \, \mathrm{d}v}{v}, \quad z_0 = \frac{\iiint\limits_D z \, \mathrm{d}v}{v}. \tag{VI}$$

We see from this that x_0, y_0, z_0 do not depend on the density.

Hence: *if the density is constant, then the position of the centre of gravity does not depend on the density.*

The same relates to surfaces and curves.

It can be shown that the theorems concerning the centre of mass for material systems of points, proved in § 2, hold also in the case of material bodies, surfaces and curves.

Geometric solids, surfaces and curves. The *statical moment of a geometric solid* is defined as the statical moment of a material body having the form of the given solid and a density $\varrho =$ const; usually we suppose that $\varrho = 1$.

The centre of mass of this body (which does not depend on ϱ) is called the *centre of gravity of the geometric solid.*

In the same manner we define the statical moment and the centre of gravity for geometric surfaces and curves.

Statical moments and centres of mass of geometric configurations are obtained, therefore, by putting $\varrho = 1$ in the given formulae (I)—(V), (1) and (2), and assuming because of this that m denotes the volume, area, or length, depending on whether the geometric configuration is a solid, surface, or curve.

We shall now become acquainted with a theorem which in many cases facilitates the finding of the centre of mass. Let us cut the given solid D by planes parallel to a certain plane Π. Let us assume that the centres of gravity of these sections lie in a certain plane σ.

Under these assumptions it can be proved that the centre of gravity of the solid D also lies in the plane σ.

This is intuitively evident. For let us cut the solid D into thin layers by means of planes parallel to the plane Π. We can assume, as an approximation, that the centre of mass of each layer lies in the plane σ. Therefore the statical moment of each layer with respect to the plane σ is equal to zero. It follows from this that the statical moment of the entire solid D with respect to the plane σ is zero (because it is equal to the sum of the moments of the separate layers). Hence the centre of mass of the solid D will also lie in the plane σ.

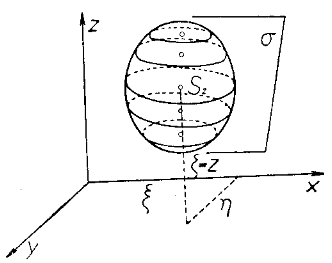

Fig. 115.

A rigorous proof can be carried out in the following manner. Let us suppose that Π is a horizontal plane, and σ is a plane having the equation

$$Ax + By + Cz + E = 0. \tag{3}$$

Let us denote the section of the solid D made by a horizontal plane at the height z by D_z (Fig. 115). Let ξ, η, and $\zeta = z$ be the coordinates of the centre of gravity S_z of the section D_z. Obviously ξ, η, and ζ are functions of z, and by (3)

$$A\xi + B\eta + C\zeta + E = 0. \tag{4}$$

Denoting by x_0, y_0, z_0 the coordinates of the centre of mass of the solid D, we obtain from (VI)

$$Ax_0 + By_0 + Cz_0 + E =$$
$$= \frac{1}{v}\Big(A\iiint_D x\,\mathrm{d}v + B\iiint_D y\,\mathrm{d}v + C\iiint_D z\,\mathrm{d}v + Ev\Big). \tag{5}$$

Let P_z be the area of the section D_z, and z' and z'' (where $z' < z''$) the limits between which z varies. Then

$$P_z = \iint_{D_z} \mathrm{d}x\,\mathrm{d}y. \tag{6}$$

Resolving the triple integral into an iterated integral, we obtain:

$$v = \iiint_D \mathrm{d}v = \int_{z'}^{z''} \mathrm{d}z \iint_{D_z} \mathrm{d}x\,\mathrm{d}y = \int_{z'}^{z''} P_z\,\mathrm{d}z, \tag{7}$$

$$\iiint_D z\,\mathrm{d}v = \int_{z'}^{z''} z\,\mathrm{d}z \iint_{D_z} \mathrm{d}x\,\mathrm{d}y = \int_{z'}^{z''} P_z z\,\mathrm{d}z = \int_{z'}^{z''} P_z \zeta\,\mathrm{d}z, \tag{8}$$

$$\iiint_D x \, dv = \int_{z'}^{z''} dz \iint_{D_z} x \, dx \, dy. \tag{9}$$

Since

$$\iint_{D_z} x \, dx \, dy$$

represents the statical moment of the section D_z with respect to the yz-plane,

$$\iint_{D_z} x \, dx \, dy = P_z \xi.$$

Therefore by (9)

$$\iiint_D x \, dv = \int_{z'}^{z''} P_z \xi \, dz, \quad \text{and similarly} \quad \iiint_D y \, dv = \int_{z'}^{z''} P_z \eta \, dz. \tag{10}$$

Hence by (6)—(10) we get

$$Ax_0 + By_0 + Cz_0 + E = \frac{1}{v} \int_{z'}^{z''} (A\xi + B\eta + C\zeta + E) \, P_z \, dz.$$

From formula (4) we obtain then

$$Ax_0 + By_0 + Cz_0 + E = 0.$$

Therefore the centre of gravity of the solid D lies in the plane σ. We have thus proved the

Theorem. *If the centres of gravity of parallel sections of a given solid lie in one plane, then the centre of gravity of this solid lies in this same plane.*

In particular, it follows that if the centres of gravity of the sections lie on one line, then the centre of gravity of this solid lies on this line. For if an arbitrary plane is passed through this line, then by the theorem just proved, the centre of gravity of the solid will lie in this plane.

Similar theorems hold for surfaces and plane figures.

Guldin's rules. Let a given curve C whose equation is $y = f(x)$, $f(x) \geqq 0$ for $a \leqq x \leqq b$, lie in the xy-plane. Denote the length of the curve by l. By (V) the centre of gravity is expressed by the formulae:

$$x_0 = M_y / m = \int_a^b x \, ds / l, \quad y_0 = M_x / m = \int_a^b y \, ds / l. \tag{11}$$

The area of the surface generated by revolving the given curve about the x-axis is

$$P = 2\pi \int_a^b y \, ds.$$

Hence by (11)

$$P = 2\pi l y_0. \tag{I.}$$

A similar formula is obtained for an arbitrary curve lying above the x-axis.

Since the centre of mass describes a circle of radius y_0 as the curve revolves, $2\pi y_0$ denotes the circumference of this circle.

Hence: *the area of a surface generated by revolving a plane curve about an axis lying in the plane of this curve and not cutting it, is equal to the product of the length of the curve and the length of the path described by the centre of gravity.*

This is the so-called *Guldin's first rule.*

Let us take under consideration for the same curve the region D bounded by the curve, the x-axis, and the ordinates $x = a$ and $x = b$. Let us denote the area of the region D by F. By (V), p. 171, the centre of gravity of the region D has the coordinates:

$$x_0 = M_y / F = (\iint_D x \, dx \, dy) / F, \quad y_0 = M_x / F = (\iint_D y \, dx \, dy) / F. \quad (12)$$

But

$$\iint_D y \, dx \, dy = \int_a^b dx(\int_0^y y \, dy) = \tfrac{1}{2} \int_a^b y^2 \, dx.$$

Therefore by (12)

$$F y_0 = \tfrac{1}{2} \int_a^b y^2 \, dx. \quad (13)$$

If the curve revolves about the x-axis, then the volume of the solid generated will be

$$V = \pi \int_a^b y^2 \, dx,$$

whence by (13)

$$V = 2\pi y_0 F. \quad \text{II.}$$

A similar formula would be obtained for an arbitrary region lying above the x-axis.

Hence: *the volume of a solid generated by revolving a plane region about an axis lying in the plane of the region and not intersecting it, is equal to the product of the area of the region and the length of the path traversed by the centre of gravity of the region.*

This is the so-called *Guldin's second rule.*

Moments of inertia and products of inertia. Proceeding as we did in connection with statical moments, we come to the definitions of moments of inertia for solids, surfaces, and curves.

If $\varrho(x, y, z)$ denotes the density of the solid, then the moments of inertia with respect to the xy, yz and zx planes are defined by the formulae:

$$I_{xy} = \iiint_D \varrho z^2 \,\mathrm{d}v, \quad I_{yz} = \iiint_D \varrho x^2 \,\mathrm{d}v, \quad I_{zx} = \iiint_D \varrho y^2 \,\mathrm{d}v, \tag{VII}$$

the moments of inertia with respect to the coordinate axes:

$$\begin{gathered} I_x = \iiint_D \varrho(y^2 + z^2) \,\mathrm{d}v, \quad I_y = \iiint_D \varrho(x^2 + z^2) \,\mathrm{d}v, \\ I_z = \iiint_D \varrho(x^2 + y^2) \,\mathrm{d}v, \end{gathered} \tag{VIII}$$

and the products of inertia with respect to the coordinate planes:

$$D_x = \iiint_D \varrho yz \,\mathrm{d}v, \quad D_y = \iiint_D \varrho zx \,\mathrm{d}v, \quad D_z = \iiint_D \varrho xy \,\mathrm{d}v. \tag{IX}$$

In order to obtain the moments of inertia of surfaces (curves), it is necessary to replace the triple integral by a double (single) integral over a surface (over a curve) and to substitute $\mathrm{d}\sigma$ ($\mathrm{d}s$) for $\mathrm{d}v$ in the given formulae just as in the case of statical moments. The definitions of the radius of gyration as well as those of a concentrated mass remain unchanged. The theorems proved for systems of material points obtain here also.

§ 8. Centres of gravity of some curves, surfaces and solids. If a line, surface, or solid has a centre of symmetry, then it is at the same time its centre of gravity. Therefore the centre of gravity of a segment, parallelogram, circle, parallelepiped, sphere and cylinder is the centre of symmetry of these configurations.

Broken line. The centre of gravity of a broken line, e. g. $ABCD$, is obtained by replacing a line segment by a material point situated at the centre of the segment, and having a mass equal to the length of the segment. The centre of gravity of this system of points will be the centre of gravity of the broken line $ABCD$ (Fig. 116).

Let d_1, d_2, d_3 denote the lengths of the segments AB, BC, CD, and $S_1(x_1, y_1)$, $S_2(x_2, y_2)$, $S_3(x_3, y_3)$ the centres of these segments. The centre of gravity of the broken line $ABCD$ will therefore have the coordinates

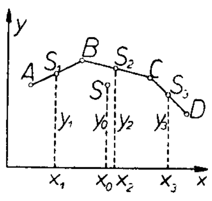

Fig. 116.

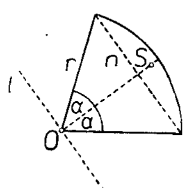

Fig. 117.

$$x_0 = \frac{d_1x_1 + d_2x_2 + d_3x_3}{d_1 + d_2 + d_3}, \quad y_0 = \frac{d_1y_1 + d_2y_2 + d_3y_3}{d_1 + d_2 + d_3}. \tag{1}$$

The arc of a circle of radius r, subtending a central angle of 2α, has the bisector of this angle as an axis of symmetry. Therefore the centre of gravity of the arc lies on this bisector (Fig. 117).

In order to determine the distance of the centre of gravity S from the centre of the circle O, we make use of Guldin's first rule. As the arc rotates about the diameter l, perpendicular to the bisector of the angle 2α, it describes a zone of area $2r\pi h$ (where h denotes the length of the chord subtended by the arc). The length of the arc is $s = 2r\alpha$, and that of the path of the centre of gravity is $2\pi \cdot OS$. Therefore $2r\pi h = 4r\pi\alpha \cdot OS$, whence $OS = h / 2\alpha$. Since $h = 2r \sin \alpha$,

$$OS = r\frac{\sin \alpha}{\alpha}. \tag{2}$$

In particular, for the semicircle $2\alpha = \pi$; consequently $OS = 2r / \pi = 0.64r$.

Triangle. Let us cut a triangle by lines parallel to one of its sides. The centres of the segments lie on the median, and hence so does the centre of gravity of the triangle.

It follows from this that the centre of gravity of the triangle lies at the point of intersection of the three medians, and hence at a distance of one third of the corresponding altitude from each side.

Trapezoid. The centres of the segments parallel to the base of a trapezoid lie on the median, and hence so does the centre of gravity S of the trapezoid.

In order to determine the distance y_0 of the centre of gravity S from the base, let us calculate the statical moment of the trapezoid with respect to the base. Let a denote the base, b the parallel side, h the altitude and P the area of the trapezoid. The statical moment with respect to the base is

$$M = Py_0 = \tfrac{1}{2}(a + b)\, hy_0. \tag{3}$$

Dividing the trapezoid into a parallelogram and a triangle, we get

$$M = bh \cdot \tfrac{1}{2}h + \tfrac{1}{2}(a - b)\, h \cdot \tfrac{1}{3}h = \tfrac{1}{6}h^2(a + 2b). \tag{4}$$

By comparing (3) and (4) we get

$$y_0 = \tfrac{1}{3} \cdot \frac{a + 2b}{a + b}\, h. \tag{5}$$

From this follows the geometric construction of the centre of gravity shown in Fig. 118. From the similarity of triangles BCS and ADS we get $(h - y_0) / y_0 = (\frac{1}{2}b + a) / (\frac{1}{2}a + b)$, from which we get y_0 in agreement with formula (5).

Polygon. In order to determine the centre of gravity of a polygon we break it up into triangles (trapezoids, rectangles), and then we compute the statical moments of the separate parts with respect to the axes of the system.

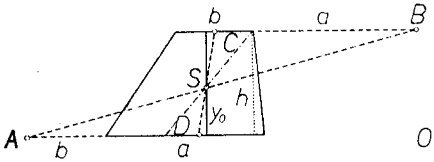

Fig. 118.

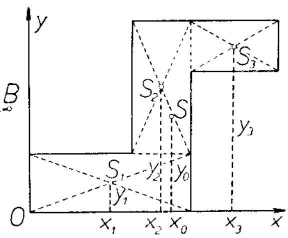

Fig. 119.

Let us denote the area of the configuration given in Fig. 119 by p. We break it up into three rectangles having areas p_1, p_2, and p_3. Let x_1, y_1, x_2, y_2 and x_3, y_3 be the coordinates of the centres of gravity with respect to the x and y axes. We have $M_x = p_1y_1 + p_2y_2 + p_3y_3$, and $M_y = p_1x_1 + p_2x_2 + p_3x_3$; hence the centre of gravity S has the coordinates

$$x_0 = M_y / p, \qquad y_0 = M_x / p. \tag{6}$$

Sector of a circle. Let us consider the sector of the circle OAB (Fig. 120). Because of symmetry the centre of gravity S of the sector lies on the bisector of the central angle 2α. The distance OS of the centre of mass from the centre of the circle O is obtained by using Guldin's second rule. The sector OAB generates a spherical sector by revolving about the radius $OA = r$. The altitude of the segment of the spherical sector will be $CA = OA - OC = r - r\cos 2\alpha = 2r\sin^2\alpha$, from which the volume of the spherical sector $v = \frac{2}{3}r^2\pi \cdot 2r\sin^2\alpha = \frac{4}{3}r^3\pi\sin^2\alpha$. The centre of gravity will describe a circle of radius $y_0 = OS \cdot \sin\alpha$. The area of the sector is $r^2\alpha$. Hence by Guldin's second rule $\frac{4}{3}r^3\pi\sin^2\alpha = 2\pi y_0 \cdot r^2\alpha = 2\pi OS\sin\alpha \cdot r^2\alpha$, whence

$$OS = \frac{2\sin\alpha}{3\alpha}\,r. \tag{7}$$

For a semicircle we have in particular $2\alpha = \pi$, whence

$$OS = 4r / 3\pi = 0.42r. \tag{8}$$

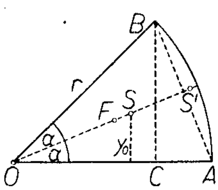

Fig. 120.

Segment of a circle. The centre of gravity S' of a segment of a circle is situated on the bisector of the central angle subtended by this segment. We obtain the distance OS' of the centre of gravity of the segment from the centre of the circle from the formula representing the statical moment of the sector OAB with respect to OA as the sum of the moments of the triangle OAB and the segment of the circle. Denoting by p the area of the sector, by p' the area of the triangle OAB, by p'' the area of the segment, and by F the centre of mass of the triangle OAB, we obtain $p \cdot OS \cdot \sin\alpha = p' \cdot OF \sin\alpha + p'' \cdot OS' \sin\alpha$. Since $p = r^2\alpha$, $p' = \frac{1}{2}r^2 \sin 2\alpha$, $p'' = p - p'$ and $OF = \frac{2}{3}r \cos\alpha$,

$$OS' = \frac{4\sin^3\alpha}{3(2\alpha - \sin 2\alpha)}\, r. \tag{9}$$

Prism. Cylinder. The centres of gravity of sections of a prism made by planes parallel to a base lie on a line joining the centres of gravity of both bases. The sections made by planes parallel to one of the lateral faces are parallelograms (or consist of several parallelograms); the centres of gravity of these sections lie in a plane parallel to the base and passing half way up the altitude. The discussion for the cylinder is similar.

It follows from this that *the centre of gravity of a prism or a cylinder lies halves the straight line joining the centres of gravity of both its bases.*

Pyramid. Cone. The centres of gravity of sections parallel to the base of a pyramid lie on a line joining the vertex with the centre of gravity of the base. Hence the centre of gravity S of the pyramid also lies on this line. In order to determine the height at which this centre of gravity lies we shall calculate the statical moment of the pyramid with respect to the plane of the base. Selecting the plane of the base as the horizontal plane, we obtain $M_{xy} = \iiint z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The region of integration extends over the entire pyramid. Let us denote the altitude of the pyramid by h (Fig. 121), the section made by a horizontal plane at a height z by D_z, and the area of the section D_z by P_z. Resolving the triple integral into an iterated integral we get

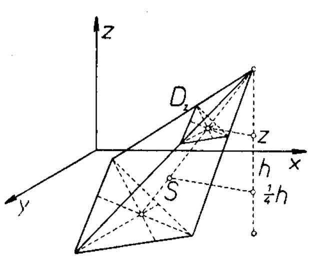

Fig. 121.

$$M_{xy} = \int_0^h z\,\mathrm{d}z \iint_{D_z} \mathrm{d}x\,\mathrm{d}y = \int_0^h z P_z \,\mathrm{d}z.$$

Let P denote the area of the base. As is known $P_z / P = (h - z)^2 / h^2$, or $P_z = [(1 - z) / h]^2 P$. Therefore

$$M_{xy} = \int_0^h z[(1 - z) / h]^2 P \, dz = \tfrac{1}{12} h^2 P.$$

On the other hand, denoting the volume of the pyramid by v and its centre of gravity by z_0, we have

$$M_{xy} = z_0 v = z_0 \cdot \tfrac{1}{3} hP.$$

By equating both formulae for M_{xy} we obtain

$$z_0 = \tfrac{1}{4} h. \tag{10}$$

The discussion for the cone is similar.

Hence: *the centre of gravity of a pyramid (cone) lies one fourth of the way up the straight line joining the vertex with the centre of gravity of the base.*

§ 9. Moments of inertia of some curves, surfaces and solids. In this § we shall assume that the curves, surfaces, and solids considered have a constant density ϱ.

Segment. Let us calculate the moment of inertia of the line segment AB of length a with respect to the line l passing through the centre O of this segment and inclined at an angle α to it (Fig. 122).

Let us suppose that AB lies on the x-axis and that O is the origin of the coordinate system. Let us subdivide the segment AB into small segments by means of the points $x_1, x_2, \ldots$ Set $\Delta x_1 = x_2 - x_1$, $\Delta x_2 = x_3 - x_2$, etc. The moment of inertia of the segment Δx_i with respect to the line l is approximately $r_i^2 \, \Delta m_i$, where Δm_i denotes the mass of the i-th segment, and r_i the distance of its left end point from l. Since $r_i = x_i \sin\alpha$, $\Delta m_i = \varrho \, \Delta x_i$, it follows that $r_i^2 \, \Delta m_i = x_i^2 \, \varrho \, \Delta x_i \sin^2\alpha$. We can say, therefore, that the moment of inertia I_l with respect to l is approximately $\Sigma x_i^2 \varrho \, \Delta x_i \sin^2\alpha$. Passing to the limit, we obtain

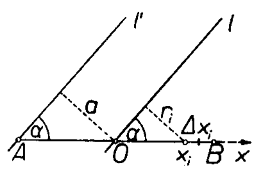

Fig. 122.

$$I_l = \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} x^2 \varrho \sin^2\alpha \, dx = \tfrac{1}{12} a^3 \varrho \sin^2\alpha.$$

The mass m of the segment AB is $m = a\varrho$. Therefore

$$I_l = \tfrac{1}{12} m a^2 \sin^2\alpha. \tag{1}$$

O is the centre of gravity of the segment AB; therefore the moment of inertia with respect to the line l' parallel to l and lying at a distance d from O is according to formula (I), p. 159, $I_{l'} = I_l + md^2$, i. e.

$$I_{l'} = \tfrac{1}{12} m (a^2 \sin^2\alpha + 12 d^2). \tag{2}$$

In particular, if l' passes through the end point A, then $d = \frac{1}{2}a \sin \alpha$, whence

$$I_{l'} = \tfrac{1}{3}ma^2 \sin^2\alpha. \tag{3}$$

If the lines l and l' are perpendicular to AB, then $\alpha = \frac{1}{2}\pi$, and the moments I_l and $I_{l'}$ are reduced to the moments of inertia with respect to the points O and A. From (1) and (3) we obtain for $\alpha = \frac{1}{2}\pi$:

$$I_o = \tfrac{1}{12}ma^2, \qquad I_A = \tfrac{1}{3}ma^2. \tag{4}$$

Rectangle. Let us pass the x and y axes of a coordinate system through the centre of a rectangle of sides a and b. Since these axes are axes of symmetry, they are at the same time central axes and therefore

$$I_y = \iint_D x^2\varrho \, \mathrm{d}x \, \mathrm{d}y = \varrho \int_{-\frac{1}{2}b}^{+\frac{1}{2}b} \mathrm{d}y \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} x^2 \, \mathrm{d}x = \tfrac{1}{12}a^3b\varrho.$$

The mass of the rectangle is $m = ab\varrho$; hence

$$I_y = \tfrac{1}{12}ma^2; \quad \text{similarly} \quad I_x = \tfrac{1}{12}mb^2. \tag{5}$$

The product of inertia D_z is zero, hence the central ellipse of inertia has the equation (p. 167, (3)) $I_x x^2 + I_y y^2 = c^2$, or $\frac{1}{12}mb^2x^2 + \frac{1}{12}ma^2y^2 = c^2$. The constant c is arbitrary; putting $c^2 = \frac{1}{12}ma^2b^2\lambda^2$, where λ is a new arbitrary constant, we obtain

$$(x / \lambda a)^2 + (y / \lambda b)^2 = 1. \tag{6}$$

Hence: *central ellipses of inertia have axes proportional to the sides of the rectangle.*

The moment of inertia with respect to the line l (Fig. 123) passing through O and making an angle α with the x-axis is (p. 166, formula (1)) $I_l = I_x \cos^2\alpha + I_y \sin^2\alpha$ or

$$I_l = \tfrac{1}{12}m(b^2 \cos^2\alpha + a^2 \sin^2\alpha). \tag{7}$$

The moments of inertia I_a and I_b with respect to the sides a and b of the rectangle are $I_a = I_x + m(\frac{1}{2}b)^2$ and $I_b = I_x + m(\frac{1}{2}a)^2$, or

$$I_a = \tfrac{1}{3}mb^2, \qquad I_b = \tfrac{1}{3}ma^2. \tag{8}$$

Square. Retaining the notation used for the rectangle, we have $a = b$, whence $I_x = I_y$. It follows from this that the central ellipse of inertia is a circle. The centre of the square is therefore a circular point.

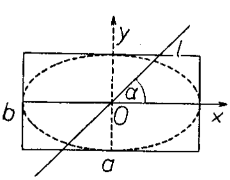

Fig. 123.

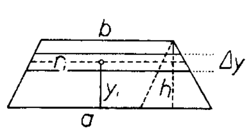

Fig. 124.

Trapezoid. In order to determine the moment of inertia of a trapezoid with respect to the base a (Fig. 124), let us divide the trapezoid into narrow strips parallel to the base. Let us denote the widths of these strips by $\Delta y_1, \Delta y_2, \ldots$, the distances of their centres from the base by $y_1, y_2, \ldots$, and the lengths of the segments passing through the centres of the strips and parallel to the base by $r_1, r_2, \ldots$ We can assume that the moment of inertia of the i-th strip with respect to the side a is approximately $\Delta m_i y_i^2$, where Δm_i denotes the mass of the i-th strip. The moment of inertia I_a with respect to the side a is approximately equal to $\Sigma \Delta m_i y_i^2$. But $\Delta m_i = \Delta y_i r_i \varrho$. From Fig. 124 we see that $(r_i - b) / (a - b) = (h - y_i) / h$, whence $r_i = a - (a - h) y_i / h$. Therefore I_a is approximately

$$\sum \left[a - (a - b) \frac{y_i}{h} \right] \varrho y_i^2 \, \Delta y_i.$$

Passing to the limit, we get

$$I_a = \int_0^h \left[a - (a - b) \frac{y}{h} \right] \varrho y^2 \, \mathrm{d}y = \tfrac{1}{12} \varrho (a + 3b) h^3.$$

Since $m = \frac{1}{2}(a + b) h\varrho$,

$$I_a = \tfrac{1}{6} \cdot \frac{a + 3b}{a + b} mh^2. \tag{9}$$

Triangle. From the last formula we obtain the moment of inertia of a triangle with respect to the base by putting $b = 0$. We get

$$I_a = \tfrac{1}{6} mh^2. \tag{10}$$

Parallelogram. Putting $b = a$ in formula (9), we obtain the moment of inertia of a parallelogram with respect to one of its sides:

$$I_a = \tfrac{1}{3} mh^2. \tag{11}$$

Rectangular parallelepiped. Let us place the origin of the coordinate system at the centre of a rectangular parallelepiped, so that the x, y and z axes be parallel to the edges, whose lengths we denote by a, b, and c. The moment of inertia with respect to the x-axis is

$$I_x = \iiint \varrho (y^2 + z^2) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \varrho \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \mathrm{d}x \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \mathrm{d}y \int_{-\frac{1}{2}c}^{\frac{1}{2}c} (y^2 + z^2) \, \mathrm{d}z =$$

$$= \tfrac{1}{12} abc\varrho (b^2 + c^2).$$

Setting $m = abc\varrho$, we obtain

$$I_x = \tfrac{1}{12} m(b^2 + c^2), \tag{12}$$

and similarly $I_y = \frac{1}{12} m(a^2 + c^2)$, $I_z = \frac{1}{12} m(a^2 + b^2)$.

The moment of inertia I_a with respect to the edge a is $I_a = I_x + md^2$, where $d = \frac{1}{2}\sqrt{b^2 + c^2}$; hence.

$$I_v = \tfrac{1}{3}m(b^2 + c^2), \tag{13}$$

and similarly $I_b = \frac{1}{3}m(a^2 + c^2)$, $I_c = \frac{1}{3}m(a^2 + b^2)$.

Circumference of a circle. The moment of inertia of the circumference of a circle of radius r with respect to the centre O is obviously

$$I_o = mr^2. \tag{14}$$

In order to determine the moment of inertia with respect to a diameter, let us choose O as the origin of the coordinate system (x, y). We obviously have $I_x = I_y$. Since $I_o = I_x + I_y$, then $I_o = 2I_x$ or $I_x = \frac{1}{2}I_o$. From this the moment of inertia with respect to a diameter is

$$I_x = \tfrac{1}{2}mr^2. \tag{15}$$

Circle. Because of symmetry the moments of inertia of a circle with respect to the diameters are equal. Let us select the centre of the circle O as the centre of the coordinate system (x, y) (Fig. 125).

Therefore $I_x = I_y$, and since the moment of inertia with respect to the centre $I_o = I_x + I_y$, then $I_o = 2I_x$ and $I_x = \frac{1}{2}I_o$. In order to calculate I_o, let us divide the circle into rings by means of concentric circles of radii $x_1, x_2, \ldots$ Let us put $\Delta x_1 = x_2 - x_1$, $\Delta x_2 = x_3 - x_2, \ldots$ We can assume that the moment of inertia of the i-th ring with respect to O is approximately $\Delta m_i x_i^2$, where Δm_i denotes the mass of this ring. The area of a ring is approximately $2\pi x_i\, \Delta x_i$; hence $\Delta m_i = 2\pi x_i \varrho\, \Delta x_i$. Therefore aproximately $I_o = \Sigma 2\pi x_i^3 \varrho \Delta x_i$. Passing to the limit, we obtain

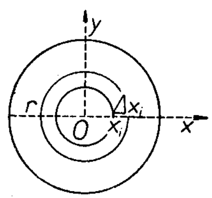

Fig. 125.

$$I_o = \int_0^r 2\pi x^3 \varrho\, \mathrm{d}x = \tfrac{1}{2}\pi\varrho r^4. \tag{16}$$

Since the mass of a circle $m = r^2\pi\varrho$,

$$I_o = \tfrac{1}{2}mr^2 \quad \text{and} \quad I_x = \tfrac{1}{4}mr^2. \tag{17}$$

Surface of a sphere. The moment of inertia of a surface of a sphere with respect to the centre is obviously

$$I_o = mr^2. \tag{18}$$

In order to determine the moment of inertia of a sphere with respect to a diameter, let us place the origin of the coordinate system at the centre of the sphere. Because of symmetry we have $I_x = I_y = I_z$. Since $2I_o = I_x + I_y + I_z$, $I_o = \frac{3}{2}I_x$. Therefore $I_x = \frac{2}{3}I_o$, whence

$$I_x = \tfrac{2}{3}mr^2. \tag{19}$$

Sphere. Taking the centre of the sphere as the origin of the coordinate system, we have because of symmetry as before $I_x = \frac{2}{3}I_o$. Let us calculate the moment I_o by proceeding as in the case of the circle, i. e. dividing the sphere into layers by means of concentric spheres. We obtain

$$I_o = \tfrac{3}{5}mr^2 \text{ and } I_x = \tfrac{2}{5}mr^2. \tag{20}$$

Cylinder of revolution. Let us denote the radius of the base by r and the altitude of the cylinder by h (Fig. 126). Let us take the centre of the axis of the cylinder as the origin of the coordinate system, and the axis of the cylinder as the z-axis.

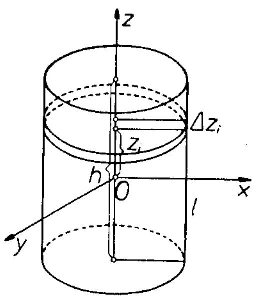

Fig. 126.

In order to calculate I_z, let us proceed as in the case of the circle, i. e. let us divide the cylinder into layers by means of cylinders whose bases are concentric with the base of the cylinder. We obtain

$$I_z = \tfrac{1}{2}mr^2. \tag{21}$$

In order to calculate I_x and I_y, let us cut the cylinder into slices by means of planes parallel to the base. Let us denote the thicknesses of the slices by $\Delta z_1, \Delta z_2, \ldots$, the coordinates of the centres of their bases by $z_1, z_2, \ldots$, and the masses of the slices by $\Delta m_1, \Delta m_2, \ldots$ The moment of inertia of the i-th slice with respect to a line parallel to the x-axis and passing through the centre of gravity of this slice is approximately equal to $\frac{1}{4}\Delta m_i r^2$ (like the moment of inertia of a circle with respect to a diameter). Hence the moment of inertia of a slice with respect to the x-axis is approximately $\frac{1}{4}\Delta m_i r^2 + \Delta m_i z_i^2$. Since $\Delta m_i = r^2\pi\, \Delta z_i \varrho$, approximately $I_x = \Sigma(\frac{1}{4}r^2 + z_i^2)\, r^2\pi\varrho\, \Delta z_i$, whence, passing to the limit, we obtain

$$I_x = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\tfrac{1}{4}r^2 + z^2)\, r^2\pi\varrho\, \mathrm{d}z = \tfrac{1}{12}r^2\pi\varrho h(3r^2 + h^2).$$

Since the mass of the cylinder is $m = r^2\pi\varrho h$,

$$I_x = \tfrac{1}{12}m(3r^2 + h^2). \tag{22}$$

On account of symmetry we obviously have $I_x = I_y$.

The moment of inertia of the cylinder with respect to the generatrix l is $I_l = I_z + mr^2$; hence

$$I_l = \tfrac{3}{2}mr^2. \tag{23}$$

The z-axis is an axis of symmetry, and hence a central axis as well. Because of symmetry the x and y axes are also central axes. Hence the ellipsoid of inertia has the equation $I_x x^2 + I_y y^2 + I_z z^2 = c^2$, whence because of (22) $\frac{1}{12}m(3r^2 + h^2)(x^2 + y^2) + \frac{1}{2}mr^2z^2 = c^2$ and hence

$$(x / r)^2 + (y / r)^2 + (z / \sqrt{\tfrac{1}{6}(3r^2 + h^2)}) = \lambda^2, \tag{24}$$

where λ^2 is an arbitrary constant.

The ellipsoid of inertia is therefore an ellipsoid of revolution. When $r\sqrt{3} = h$, the ellipsoid is a sphere.

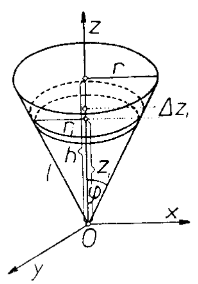

Fig. 127.

Cone of revolution. Let us denote the radius of the base by r and the altitude of the cone by h. Let us place the origin O of the coordinate system at the vertex of the cone, and let us take the axis of the cone as the z-axis (Fig. 127).

As an axis of symmetry it is also a central axis of inertia, and hence by theorem 2°, p. 165, the principal axis of inertia at the point O. Because of symmetry the x and y axes are also principal axes of inertia at the point O.

Let us cut the cone into slices of thickness Δz_i be means of planes parallel to the base. The moment of inertia of the i-th slice with respect to the z-axis is approximately $\Delta m_i r_i^2 / 2$ (like the moment of inertia of a cylinder with respect to the axis), where r_i denotes the radius of the lower base of the i-th slice. Let z_i denote the coordinate of the centre of the lower base of the i-th slice; then $r_i / r = z_i / h$, whence

$$r_i = rz_i / h. \tag{25}$$

Since $\Delta m_i = r_i^2\pi\, \Delta z_i \varrho$ approximately, by (25) we have

$$\tfrac{1}{2}\Delta m_i r_i^2 = \tfrac{1}{2}(r / h)^4\, \pi\varrho z_i^4\, \Delta z_i, \tag{26}$$

whence $I_z = \Sigma\frac{1}{2}(r / h)^4\, \pi\varrho z_i^4\, \Delta z_i$ approximately. Passing to the limit, we obtain

$$I_z = \int_0^h \tfrac{1}{2}(r / h)^4\, \pi\varrho z^4\, \mathrm{d}z = \tfrac{1}{10}r^4h\pi\varrho.$$

The mass of the cone is $m = \frac{1}{3}r^2\pi h\varrho$; hence

$$I_z = \tfrac{3}{10}mr^2. \tag{27}$$

In order to calculate I_x, let us note that the moment of inertia of the i-th slice with respect to a line parallel to the x-axis and passing through the centre of gravity of this slice is approximately $\frac{1}{4}\Delta m_i r_i^2$. Therefore with respect to the x-axis it is $\frac{1}{4}\Delta m_i r_i^2 + \Delta m_i z_i^2$. By (25) and (26) this sum is equal to $\frac{1}{4}(r / h)^2 \pi \varrho z_i^4(4 + (r / h)^2)\, \Delta z_i = \Delta w_i$, or I_x is approximately equal to $\Sigma \Delta w_i$. Passing to the limit, we obtain

$$I_x = \int_0^h [\tfrac{1}{4}(r / h)^2 \pi \varrho z^4(4 + (r / h)^2)]\, \mathrm{d}z = \tfrac{1}{20} r^2 \pi \varrho (4h^2 + r^2)\, h,$$

whence

$$I_x = \tfrac{3}{20} m(r^2 + 4h^2). \tag{28}$$

Obviously $I_x = I_y$.

Let φ denote the angle between the z-axis and the generatrix l (lying in the xz-plane). Since the x, y, z axes are principal axes of inertia at O, by formula (I), p. 162, $I_l = I_x \cos^2\alpha + I_y \cos^2\beta + I_z \cos^2\gamma$, where α, β, γ denote the angles between the generatrix l and the axes of the coordinate system. We have $\alpha = \frac{1}{2}\pi - \varphi$, $\beta = \frac{1}{2}\pi$, and $\gamma = \varphi$, whence

$$I_l = I_x \sin^2\varphi + I_z \cos^2\varphi,$$

and hence by (27) and (28)

$$I_l = \tfrac{3}{20} m[(r^2 + 4h^2)\sin^2\varphi + 2r^2\cos^2\varphi]. \tag{29}$$

As $\tan\varphi = r / h$, we get

$$I_l = \frac{3m}{20} \cdot \frac{r^2 + 6h^2}{r^2 + h^2} r^2. \tag{30}$$