CHAPTER VIII

DYNAMICS OF A RIGID BODY

§ 1. Work and kinetic energy. In dynamics, as in statics, we shall also frequently assume that a rigid body is a rigid system of material points. As a result we shall be able to apply to a rigid body the theorems from dynamics concerning a system of material points.

Dynamical magnitudes. Dynamical magnitudes such as momentum, kinetic energy, angular momentum, etc., which we have met in connection with systems of material points, are also defined for rigid bodies by passing to the limit as was done in defining centres of gravity, statical moments and moments of inertia, of systems of material points. (*vide* chapt. IV, p. 169). For instance, in order to define the momentum of a body, we divide the body into parallelepipeds of volumes $\Delta\tau_1, \Delta\tau_2, \ldots$, and in each one of them we next consider, one at a time, the points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, ... Let $\varrho_1, \varrho_2, \ldots$ denote the densities of the body at the chosen points, and $\mathbf{v}_1, \mathbf{v}_2, \ldots$ the velocities of these points. The masses of the parallelepipeds are approximately $m_1 = \varrho_1\,\Delta\tau_1$, $m_2 = \varrho_2\,\Delta\tau_2, \ldots$ If the body is replaced by a system of material points of masses $m_1, m_2, \ldots$ placed at $A_1, A_2, \ldots$ and having velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots$, then the momentum of this system will be equal to

$$\boldsymbol{H} = \Sigma m_i\mathbf{v}_i = \Sigma\varrho_i\mathbf{v}_i\,\Delta\tau_i. \tag{1}$$

The limit of this sum as the dimensions of the parallelepipeds tend to zero is called the *momentum of the body*.

Denoting by $\varrho(x, y, z)$ the density, by $\mathbf{v}(x, y, z)$ the velocity of a point whose coordinates are x, y, z, and by $\boldsymbol{H}$ the momentum of the body, we get:

$$H_x = \iiint_D \varrho v_x\,\mathrm{d}\tau, \quad H_y = \iiint_D \varrho v_y\,\mathrm{d}\tau, \quad H_z = \iiint_D \varrho v_z\,\mathrm{d}\tau, \tag{2}$$

where D denotes the region of space occupied by the body. We write the preceding formulae in vector form

$$\boldsymbol{H} = \iiint_D \varrho\mathbf{v}\,\mathrm{d}\tau. \tag{3}$$

Proceeding similarly, we obtain the formula

$$E = \tfrac{1}{2}\iiint\limits_D \varrho v^2 \,\mathrm{d}\tau \tag{4}$$

for the kinetic energy, where $v = |\mathbf{v}|$.

The angular momentum ((I′), p. 199) with respect to the origin of the coordinate system has the projections:

$$K_x = \iiint\limits_D \varrho(v_y z - v_z y)\,\mathrm{d}\tau, \quad K_y = \iiint\limits_D \varrho(v_z x - v_x z)\,\mathrm{d}\tau, \tag{5}$$

$$K_z = \iiint\limits_D \varrho(v_x y - v_y x)\,\mathrm{d}\tau.$$

Denoting by $\mathbf{r}$ the radius vector $\overline{OA}$, where A has the coordinates x, y, z, we can write formulae (5) in the vector form:

$$\mathbf{K} = \iiint\limits_D \varrho(\mathbf{v} \times \mathbf{r})\,\mathrm{d}\tau. \tag{6}$$

Work. Let a force $\mathbf{P}$, whose origin is at the point A, act on a rigid body, and let $\mathbf{v}$ denote the velocity of this point.

The work of the force $\mathbf{P}$ in the time from t' to t'' is expressed by the formula (p. 95)

$$L = \int\limits_{t'}^{t''} (\mathbf{P}\mathbf{v})\,\mathrm{d}t. \tag{7}$$

Let us consider an arbitrary point O in the body. The instantaneous motion of the body can be considered as the composition of an advancing motion with a velocity $\mathbf{u}$ of the point O, and a rotation with an angular velocity $\boldsymbol{\omega}$ about an axis l passing through O. The velocity of the point A is therefore ((I), p. 333)

$$\mathbf{v} = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega}, \tag{8}$$

whence

$$\mathbf{P}\mathbf{v} = \mathbf{P}\mathbf{u} + \mathbf{P}(\overline{OA} \times \boldsymbol{\omega}). \tag{9}$$

From formula (II), p. 13, we have (putting $\mathbf{a} = \mathbf{P}$, $\mathbf{b} = \overline{OA}$, and $\mathbf{c} = \boldsymbol{\omega}$)

$$\mathbf{P}(\overline{OA} \times \boldsymbol{\omega}) = \boldsymbol{\omega}(\mathbf{P} \times \overline{OA}). \tag{10}$$

Since $\mathrm{Mom}_O \mathbf{P} = \mathbf{P} \times \overline{OA}$ (p. 16), by (9) and (10)

$$\mathbf{P}\mathbf{v} = \mathbf{P}\mathbf{u} + \boldsymbol{\omega}\,\mathrm{Mom}_O \mathbf{P},$$

whence by (7)

$$L = \int\limits_{t'}^{t''} \mathbf{P}\mathbf{u}\,\mathrm{d}t + \int\limits_{t'}^{t''} \boldsymbol{\omega}\,\mathrm{Mom}_O \mathbf{P}\,\mathrm{d}t. \tag{11}$$

If the forces $\mathbf{P}_1, \mathbf{P}_2 \ldots$ act on a body, then their work is according to (11)

$$L = \int_{t'}^{t''} (\Sigma \boldsymbol{P}_i)\, \boldsymbol{u}\, \mathrm{d}t + \int_{t'}^{t''} \boldsymbol{\omega} (\Sigma \mathrm{Mom}_O \boldsymbol{P}_i)\, \mathrm{d}t.$$

Denoting by $\boldsymbol{P}$ the sum of the forces and by $\boldsymbol{M}_O$ the total moment with respect to O we get

$$L = \int_{t'}^{t''} (\boldsymbol{P}\boldsymbol{u})\, \mathrm{d}t + \int_{t'}^{t''} (\boldsymbol{\omega} \boldsymbol{M}_O)\, \mathrm{d}t. \tag{I}$$

From this formula it follows that *equipollent systems of forces do equal work.*

In particular, *the work done by a system of forces equipollent to zero* ($\boldsymbol{P} = 0$, $\boldsymbol{M} = 0$) *is zero.*

Denoting by α the angle which $\boldsymbol{M}_O$ makes with the axis l of instantaneous rotation, and by ω the component of the vector $\boldsymbol{\omega}$ with respect to the axis l, we have $\boldsymbol{\omega}\boldsymbol{M}_O = \omega |\boldsymbol{M}_O| \cos\alpha$. But $|\boldsymbol{M}_O| \cos\alpha$ is the projection of the moment $\boldsymbol{M}_O$ on the axis l. Consequently $|\boldsymbol{M}_O| \cos\alpha$ is equal to M_l, i. e. to the total moment of the forces with respect to the instantaneous axis of rotation; hence from (I) we get

$$L = \int_{t'}^{t''} (\boldsymbol{P}\boldsymbol{u})\, \mathrm{d}t + \int_{t'}^{t''} \omega M_l\, \mathrm{d}t. \tag{I'}$$

When a body moves with an advancing motion, then $\boldsymbol{\omega} = 0$; therefore by (I)

$$L = \int_{t'}^{t''} (\boldsymbol{P}\boldsymbol{u})\, \mathrm{d}t. \tag{12}$$

When a body rotates about a point O, then $\boldsymbol{u} = 0$; hence by (I′)

$$L = \int_{t'}^{t''} \omega M_l\, \mathrm{d}t. \tag{13}$$

Formula (13) also holds when a body rotates about a fixed axis l; we obtain it from formula (I′) by choosing the point O on the axis l. Denoting the angle of rotation in this case by φ, we get $\omega = \mathrm{d}\varphi / \mathrm{d}t$, whence $\omega\, \mathrm{d}t = \mathrm{d}\varphi$; hence by (13)

$$L = \int_{\varphi'}^{\varphi''} M_l\, \mathrm{d}\varphi, \tag{14}$$

where φ' and φ'' denote the angles at the initial and final positions of the body.

Kinetic energy. As we know (p. 331), the instantaneous motion of a body with respect to an arbitrary point O of the body is a rotation with an instantaneous angular velocity $\boldsymbol{\omega}$ about an axis passing through O. Let I denote the moment of inertia with respect to the instantaneous axis of rotation. The kinetic energy of the relative motion is $\frac{1}{2}I\omega^2$. Therefore by König's theorem (p. 215) the kinetic energy of the body is expressed by the formula

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}I\omega^2 + m\mathbf{u}(\mathbf{v}_0 - \mathbf{u}), \tag{II}$$

where u denotes the absolute value of the velocity $\mathbf{u}$ of the point O, $\mathbf{v}_0$ the velocity of the centre of mass, and m the mass of the body.

Therefore: *the kinetic energy of a rigid body is equal to the sum of:*

1. *the kinetic energy of an advancing motion with a velocity of an arbitrary point O of the body,*

2. *the kinetic energy of an instantaneous rotation with respect to an instantaneous axis passing through the point O and*

3. *the scalar product* $m\mathbf{u}(\mathbf{v}_0 - \mathbf{u})$,

where m denotes the mass of the body, $\mathbf{u}$ the velocity of the point O, and $\mathbf{v}_0$ the velocity of the centre of mass.

If the centre of mass is chosen as the point O, then $\mathbf{u} = \mathbf{v}_0$; putting $v_0 = |\mathbf{v}_0|$, we consequently get

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I\omega^2. \tag{II'}$$

Therefore: *the kinetic energy of a rigid body is equal to the sum of the kinetic energy of an advancing motion with a velocity of the centre of mass and of the kinetic energy of an instantaneous rotation with respect to an instantaneous axis passing through the centre of mass.*

If the instantaneous motion is an instantaneous twist about the central axis passing through O, then the vectors $\boldsymbol{\omega}$ and $\mathbf{u}$ are parallel. The velocity $\mathbf{v}_0$ of the centre of mass S is then $\mathbf{v}_0 = \mathbf{u} + \overline{OS} \times \boldsymbol{\omega}$. Since $\mathbf{v}_0 - \mathbf{u} = \overline{OS} \times \boldsymbol{\omega}$, the vector $\mathbf{v}_0 - \mathbf{u}$ is perpendicular to $\boldsymbol{\omega}$ and therefore also to $\mathbf{u}$. It follows from this that the scalar product $\mathbf{u}(\mathbf{v}_0 - \mathbf{u})$ is zero. By (II) we consequently have

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}I\omega^2. \tag{15}$$

Therefore: *if the instantaneous motion of a rigid body is represented as a twist, then the kinetic energy is the sum of the kinetic energies of the advancing and rotational motions.*

An instantaneous plane motion is an instantaneous rotation about the instantaneous centre of rotation with an instantaneous angular velocity.

The kinetic energy in a plane motion is $E = \frac{1}{2}I\omega^2$, where I is the moment of inertia with respect to the instantaneous centre of rotation, and ω the instantaneous angular velocity.

§ 2. Equations of motion. Motion of the centre of mass. Let m denote the mass of the body, $\boldsymbol{p}_0$ the acceleration of the centre of mass, and $\boldsymbol{P}$ the sum of the external forces acting on the body. Then (p. 196)

$$m\boldsymbol{p}_0 = \boldsymbol{P}. \tag{I}$$

Hence, knowing the sum of the external forces, we can determine the motion of the centre of mass of the body.

Principle of angular momentum. Let $\boldsymbol{K}$ denote the angular momentum, $\boldsymbol{M}$ the total moment of the external forces with respect to a fixed point or with respect to the centre of mass. Then (p. 202)

$$\boldsymbol{K}^{\cdot} = \boldsymbol{M}. \tag{II}$$

Therefore, knowing the total moment of the forces with respect to a fixed point or with respect to the centre of mass, we can determine the angular momentum.

If we calculate the acceleration $\boldsymbol{p}_0$ of the centre of mass and the angular momentum $\boldsymbol{K}$ from equations (I) and (II), the motion of the body will be determined. For having $\boldsymbol{p}_0$ given, we can determine the motion of the centre of mass. And knowing the angular momentum $\boldsymbol{K}$, we can (as we shall show later, p. 394) determine the instantaneous angular velocity $\boldsymbol{\omega}$. Since the motion of one point and the instantaneous angular momentum define the motion of a body (p. 337), equations (I) and (II) are sufficient to determine this motion.

Principle of kinetic energy. Let us consider a rigid body as a rigid system of material points. Then by the theorem given on p. 208 the internal forces do no work. Consequently, only the external forces do work. From the theorem on kinetic energy (p. 216) it follows that *the increase in kinetic energy of a rigid body is equal to the work of the external forces.*

If the external forces possess a potential, then (p. 216) *the sum of the kinetic and potential energies is constant.*

D'Alembert's principle. As we know (p. 188), the forces of inertia balance the forces acting on the points of a system. Since the internal forces have a sum and total moment equal to zero, *the forces of inertia balance the external forces.*

This principle reduces the investigation of the motion of a rigid body to problems of statics.

Advancing motion of a body. If the instantaneous motion of a rigid body is an advancing motion, then the angular momentum with respect to the centre of mass is zero (p. 200). Conversely:

If the angular momentum with respect to the centre of mass is zero at some instant, then the instantaneous motion of a rigid body is an advancing motion.

Proof. Let us assume that the angular momentum $\mathbf{K}$ with respect to the centre of mass is zero. The instantaneous motion of the rigid body can be considered as the composition of an instantaneous advancing motion with a velocity of the centre of mass and a rotation with an angular velocity ω about an axis l passing through the centre of mass. Since $\mathbf{K}$ is the sum of the angular momenta of the advancing and rotational motions, $\mathbf{K}$ is equal to the angular momentum of the rotational motion because the angular momentum of the advancing motion with respect to the centre of mass is zero (p. 200).

Let us denote by K the angular momentum with respect to the instantaneous axis of rotation l. From formula (7), p. 201, we have $K = I\omega$, where I is the moment of inertia with respect to the axis l. Since K is the projection of the angular momentum $\mathbf{K}$ on the instantaneous axis of rotation, $K = 0$. Consequently $I\omega = 0$, whence $\omega = 0$, q. e. d.

If a body moves with an advancing motion during a certain interval of time, then the angular momentum $\mathbf{K}$ with respect to the centre of mass is constantly zero during this time. Because of this the derivative $\mathbf{K}^{\cdot}$ of the angular momentum is also zero. From formula (II), p. 364, it follows that $\mathbf{M} = 0$, which means that the moment of the forces with respect to the centre of mass is zero. Hence by theorem 1, p. 26, the forces have a resultant acting at the centre of mass.

Conversely, if the forces have a resultant acting constantly at the centre of mass, then $\mathbf{M} = 0$; hence $\mathbf{K}^{\cdot} = 0$ constantly, i. e. $\mathbf{K} = \text{const}$. If we assume that the instantaneous motion at the initial moment was an advancing motion, i. e. that $\mathbf{K} = 0$ at that moment, then $\mathbf{K} = 0$ constantly, which means that the body will move with an advancing motion.

Therefore: *in order that a body move with an advancing motion, it is necessary and sufficient that the following conditions be satisfied:*

1° *the instantaneous motion is an instantaneous advancing motion at the initial moment,*

2° *the forces have a resultant acting at the centre of mass at each moment.*

Conditions of equilibrium. The necessary and sufficient conditions which must be satisfied by a system of forces in equilibrium follow easily from conditions 1° and 2° (cf. p. 244).

If a body is at rest, then the acceleration $\boldsymbol{p}_0$ of the centre of mass and the angular momentum $\boldsymbol{K}$ are equal to zero; consequently by (I) and (II), p. 364, $\boldsymbol{P} = 0$ and $\boldsymbol{M} = 0$.

Conversely, if we assume that the body was at rest at $t = t_0$ and that $\boldsymbol{P} = 0$ and $\boldsymbol{M} = 0$ constantly, then from conditions 1° and 2° it follows that the body will move with an advancing motion. The centre of mass will be at rest because $\boldsymbol{p}_0 = 0$ and the initial velocity $\boldsymbol{v}_0$ is zero; the whole body will consequently be at rest. The system of forces is therefore in equilibrium.

Hence we have proved that *the necessary and sufficient condition for the equilibrium of forces is the vanishing of the sum and of the total moment of the forces.*

Reactions of bodies in contact. Two rigid bodies I and II, in contact at the point A, act on each other with certain forces subject to the law of action and reaction. The forces with which body II acts on body I can be replaced by one force $\boldsymbol{R}$ with its origin at A and a force couple of moment $\boldsymbol{M}$.

At present we do not possess a general theory for the moment $\boldsymbol{M}$. Only in some particular cases have certain laws been established concerning $\boldsymbol{M}$. For the force $\boldsymbol{R}$, however, experiments have yielded rather general laws (although approximate), which give sufficiently accurate results in practice.

Let us assume that the bodies have a common tangent plane Π at A. The component vector $\boldsymbol{N}$ (of the reaction $\boldsymbol{R}$) perpendicular to Π is called the *normal reaction*, and the tangential component $\boldsymbol{T}$ the *friction*.

Experiments show that certain relations between $\boldsymbol{N}$ and $\boldsymbol{T}$ obtain. We shall consider two cases:

1° The poins of contact of the two bodies have equal velocities: these points therefore have a zero velocity relative to each other. The instantaneous motion of one body relative to the other is a rotation about an axis passing through the point of contact, it is consequently a relative rolling motion. In this case (putting $T = |\boldsymbol{T}|$ and $N = |\boldsymbol{N}|$), we have

$$T \leqq fN, \tag{1}$$

where f denotes the coefficient of static friction (cf. p. 268), depending only on the nature of the surfaces of two bodies at the point of contact.

2° The velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the points of contact of the two bodies are different. The relative velocity $\boldsymbol{v}_r$ of the point of contact of body I with respect to body II is $\boldsymbol{v}_r = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and lies in the tangent plane. (Fig.

283). In this case the friction has the direction of the relative velocity $\mathbf{v}_r$, but an opposite sense. Moreover, we have

$$T = \mu N, \tag{2}$$

where μ denotes the so-called *coefficient of dynamic friction*, depending only on the nature of the surfaces of the body and not on the velocities of the points. We can therefore assume that $\mu = \text{const}$ during motion as long as the bodies are in contact and as long as the points of contact have different velocities.

In general μ is somewhat smaller than f.

The laws given in 1° and 2° are approximate.

If the friction is zero the surfaces of the bodies are said to be *smooth*.

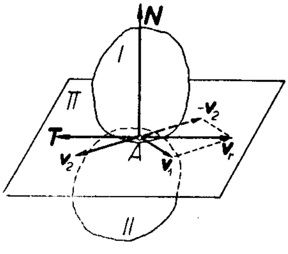

Fig. 283.

The surfaces of bodies are said to be *perfectly rough* if the bodies can move only in such a way that their points of contact have equal velocities. The relative motion of one body with respect to the other is then a rolling motion.

Work of the friction. Let $\mathbf{R}$ denote the reaction of body II on body I; then $-\mathbf{R}$ denotes the reaction of body I on body II. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the velocities of the material points of contact of the bodies. The work which the reactions at the point of contact do in the time from t' to t'' is

$$L = \int_{t'}^{t''} \mathbf{R}\mathbf{v}_1 \, dt - \int_{t'}^{t''} \mathbf{R}\mathbf{v}_2 \, dt = \int_{t'}^{t''} \mathbf{R}(\mathbf{v}_1 - \mathbf{v}_2) \, dt.$$

Since $\mathbf{v}_1 - \mathbf{v}_2$ lies in a tangent plane (or is zero), the normal reactions do no work because $\mathbf{N}(\mathbf{v}_1 - \mathbf{v}_2) = 0$. The work of the reactions is therefore reduced to the work of the forces of friction. Consequently

$$L = \int_{t'}^{t''} \mathbf{T}(\mathbf{v}_1 - \mathbf{v}_2) \, dt.$$

When $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2 \neq 0$, the friction $\mathbf{T}$ has in virtue of 2° the direction of $\mathbf{v}_r$, but an opposite sense; hence the scalar product $\mathbf{T}(\mathbf{v}_1 - \mathbf{v}_2)$ is negative. When $\mathbf{v}_1 - \mathbf{v}_2 = 0$, then $\mathbf{T}(\mathbf{v}_1 - \mathbf{v}_2) = 0$. Therefore in both cases $\mathbf{T}(\mathbf{v}_1 - \mathbf{v}_2) \leq 0$, whence $L \leq 0$.

When the motion of the bodies relative to each other is a rolling motion, then the work of the friction is zero, and when this motion is a sliding motion, then the work of the friction is negative and causes a decrease in the kinetic energy of the bodies.

Example I. A circular disk falls under the influence of its own weight in a vertical plane along a circle K. Determine the motion of the disk.

Let O denote the centre of the circle K and O' the centre of the disk, R and r their radii, m the mass of the disk, I its moment of inertia with respect to the centre of mass O', and finally φ the angle between the vertical and the segment OO' (Fig. 284).

We shall first consider the case when the disk and the circle K are smooth, and then when they are perfectly rough.

1° Let us assume that the disk as well as the circle are smooth and that at the initial moment $t = 0$ the disk was at rest. The forces acting on the disk during motion are: the weight **Q** with its initial point at the centre of mass O' and the reaction **N** whose direction passes through O'. The moment of these forces with respect to the centre of mass is therefore constantly zero. Since the disk was initially at rest, it will move with an advancing motion (p. 365), i. e. it will slide along the circle K (p. 337). The centre O' of the disk will therefore move along a circle with centre at O and of radius $a = R - r$. Consequently all the points of the disk will move along circles of radius a (cf., e. g. the path of the point A shown Fig. 284).

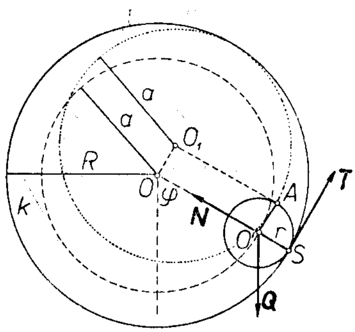

Fig. 284.

Denoting by $\boldsymbol{p}_0$ the acceleration of the centre of mass of the disk, we have by the theorem on the motion of the centre of mass (p. 364)

$$m\boldsymbol{p}_0 = \mathbf{Q} + \mathbf{N}. \tag{3}$$

Forming the projections on the tangent and normal to the path at the point O', we get:

$$ma\varphi^{\cdot\cdot} = -mg\sin\varphi, \quad ma\varphi^{\cdot 2} = N, \tag{4}$$

where $N = |\mathbf{N}|$. The first of the equations (4) can be written in the form

$$\varphi^{\cdot\cdot} = -\frac{g}{a}\sin\varphi. \tag{5}$$

Comparing equation (5) with the equation of the simple pendulum $\varphi^{\cdot\cdot} = -\frac{g}{a}\sin\varphi$ (p. 130), we see that the centre of mass of the disk will execute an oscillatory motion like that of a simple pendulum of length $l = a$.

2° Let us now assume that the disk and circle are perfectly rough. The disk will therefore (p. 367) roll along the circle K.

Let us denote by ω the instantaneous angular velocity of the disk. The kinetic energy of the disk is ((II′), p. 363)

$$E = \tfrac{1}{2}ma^2\varphi^{\cdot 2} + \tfrac{1}{2}I\omega^2$$

(because $a\varphi^{\cdot}$ is the velocity of the centre of mass). If the body is at rest at $t = 0$ and $\varphi = \bar{\varphi}_0$, then from the principle of the equivalence of work and kinetic energy (p. 364) we get

$$\tfrac{1}{2}ma^2\varphi^{\cdot 2} + \tfrac{1}{2}I\omega^2 = mga(\cos\varphi - \cos\varphi_0), \tag{6}$$

because the friction (p. 367) and the reaction $\boldsymbol{N}$ do no work.

The velocity v of the point of contact S is zero. Consequently $v = a\varphi^{\cdot} - r\omega = 0$, whence $\omega = a\varphi^{\cdot} / r$. Substituting in (6), we therefore get

$$\tfrac{1}{2}a^2(m + I / r^2)\,\varphi^{\cdot 2} = mga(\cos\varphi - \cos\varphi_0). \tag{7}$$

From this we obtain $\varphi^{\cdot}$ in terms of φ and then ω. Differentiating equation (7), we get $a^2[m + I / r^2]\,\varphi^{\cdot}\varphi^{\cdot\cdot} = - mga\varphi^{\cdot}\sin\varphi$, whence after simplifying

$$\varphi^{\cdot\cdot} = -\frac{mg}{a(m + L / r^2)}\sin\varphi. \tag{8}$$

Comparing equation (8) with the equation of the simple pendulum $\varphi^{\cdot\cdot} = -\dfrac{g}{l}\sin\varphi$ (p. 130), we see that the centre of the disk will move like a pendulum of length $l = a(1 + I / mr^2)$.

Since $I = \tfrac{1}{2}mr^2$ for a homogeneous circle, $l = \tfrac{3}{2}a$. The period of oscillation will therefore be longer than that of a pendulum of length a.

Example 2. A heavy rigid body hangs on a horizontal axis l (Fig. 285) about which it can only rotate. The position of the body is determined by giving the position of the axis l and the angle φ which the line SG, passing through the centre of mass S and cutting the axis l at right angles in the point G, makes with the vertical. The axis l cuts the vertical axis k in the point O and rotates about it with a constant angular velocity ω. What is the relation between ω and φ if φ is constant?

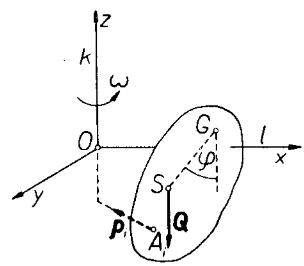

Fig. 285.

We shall solve the problem by d'Alembert's principle (p. 364). The forces of inetia balance the acting forces. Consequently the

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total moment of the forces of inertia and that of the acting forces with respect to the axis l is zero.

At a certain time t let us choose a coordinate system, taking the axis l as the x-axis and the axis k as the z-axis. Since φ = const, each point of the body moves with an angular velocity ω along a horizontal circle (whose centre lies on the z-axis). The acceleration of each point of the body is therefore directed towards the centre of the circle and is $r\omega^2$, where r denotes the radius of the circle.

Let us assume that the body is a set of material points. If $\mathbf{p}_i$ denotes the acceleration of the point A_i of mass m_i and coordinates x_i, y_i, z_i, then

$$p_{i_x} = -x_i\omega^2, \quad p_{i_y} = -y_i\omega^2, \quad p_{i_z} = 0. \tag{9}$$

Consequently the forces of inertia of the point A have the projections:

$$-m_i p_{i_x} = m_i x_i \omega^2, \qquad -m_i p_{i_y} = m_i y_i \omega^2, \qquad -m_i p_{i_z} = 0.$$

The moment of the forces of inertia with respect to the axis l (i. e. the x-axis) is:

$$B_x = \Sigma(-m_i p_{i_y})\, z_i = \omega^2\, \Sigma m_i y_i z_i = \omega^2 D_x, \tag{10}$$

where D_x denotes the product of inertia with respect to the planes xy and xz.

The centre of gravity S has the coordinates $x_0 = OG$, $y_0 = l_0 \sin\varphi$, $z_0 = -l_0\cos\varphi$ (where $l_0 = SG$). The moment of the weight with respect to the x-axis is

$$M_x = mgl_0 \sin\varphi, \tag{11}$$

where m denotes the mass of the body.

The moment of the reactions with respect to the x-axis is zero, because the reactions have their points of application on the z-axis. Therefore $B_x + M_x = 0$; hence by (10) and (11)

$$\omega^2 D_x + mgl_0 \sin\varphi = 0. \tag{12}$$

This equation is the sought for relation and can be satisfied only when $D_x \leqq 0$ (e. g., when the body is in the quadrant in which $y > 0$ and $z < 0$).

Example 3. A horizontal rod OA is attached rigidly at the point O on a vertical axis which is fixed at the points K and L (Fig. 286). A material point (a small sphere) B, which is strung on the rod, is capable of moving freely along the rod. At the initial moment $t = 0$ the rod OA

revolves about KL with an angular velocity ω_0, while the point B has a zero velocity relative to the rod and is situated at a distance x_0 from O. Determine the motion of the rod OA and of the material point B.

Let I denote the moment of inertia of the rod with respect to the axis KL, m the mass of the point B, ω the angular velocity of the rod, and x the length of the segment OB.

Let us assume that there is no friction. The external forces acting on the system consisting of the axis KL, the rod OA, and the point m, are: the reactions at L and K as well as the force of gravity. The moment of these forces with respect to KL is zero. The angular momentum of the system with respect to the axis KL is therefore constant. The angular momentum of the rod with respect to KL is $I\omega$ (p. 201).

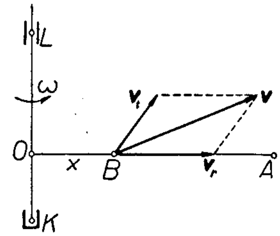

Fig. 286.

The velocity $\mathbf{v}$ of the point B is the sum of its relative velocity $\mathbf{v}_r$ with respect to the rod and the velocity of transport $\mathbf{v}_t$. The relative velocity has the component (with respect to OA) $v_r = x^{\cdot}$ and the direction OA; its moment with respect to KL is therefore zero. The velocity of transport is perpendicular to OA, has a horizontal direction, and $|\mathbf{v}_t| = x\omega$. Consequently the moment of momentum of the point B with respect to KL is equal to $mx^2\omega$. The angular momentum of the entire system is therefore $I\omega + mx^2\omega$, whence

$$(I + mx^2)\,\omega = k = \text{const.} \tag{13}$$

From the given initial conditions we have $k = (I + mx_0^2)\,\omega_0$. Let us note that the work of the acting forces is zero. Therefore the kinetic energy of the system is constant. The kinetic energy of the rod is $\frac{1}{2}I\omega^2$ (p. 363) and that of the point B

$$\tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(\mathbf{v}_r^2 + \mathbf{v}_t^2) = \tfrac{1}{2}m(x^{\cdot 2} + x^2\omega^2).$$

Consequently

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}m(x^{\cdot 2} + x^2\omega^2) = h = \text{const.} \tag{14}$$

Calculating ω from (13) and substituting in (14), we obtain

$$k^2 \,/\, (I + mx^2) + mx^{\cdot 2} = 2h. \tag{15}$$

Equation (15) defines the relative motion of the point along the rod. Knowing x, we determine ω from (13).

If the point reaches A and then leaves the rod, then the rod will revolve with a constant angular velocity ω' which is obtained from (13) by putting $x = OA = l$. The velocity of the point at the moment it leaves the rod is $v^2 = l^2\omega'^2 + \dot{x}^2$, where $\dot{x}^2$ is obtained from (15) by putting $x = l$.

Example 4. A material point M rolls down the hypotenuse of a material right-angled triangle ABC (Fig. 287). The triangle lies in a vertical plane and rests on a smooth horizontal line l. Determine the motion of the system consisting of the point M and the triangle ABC.

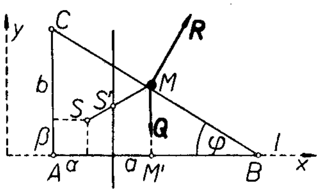

Fig. 287.

Let us take the axis l as the x-axis of the coordinate system. Let a and b denote the lengths of the legs of the triangle, S the centre of its mass, α and β the projections of the segment AS on the legs, φ the angle whose vertex is at B, m' the mass of the triangle, m'' the mass of the point, x_1 and $y_1 = \beta$ the coordinates of the point S, and x_2, y_2, the coordinates of the material point M.

Let us assume that the system consisting of the triangle and the point M was at rest at $t = 0$.

The external forces acting on the system are the reaction of the line l as well as the weight of the triangle and that of the point M. These forces have a constant vertical direction. Consequently their sum also has a constant vertical direction. From the principle of centre of mass (p. 364) it follows that the centre of mass S' of the whole system will move along a vertical (since the initial velocity was equal to zero). The coordinates of the centre of mass S' of the system are:

$$x_0 = (m'x_1 + m''x_2) \,/\, m, \quad y_0 = (m'\beta + m''y_2) \,/\, m, \tag{16}$$

where $m = m' + m''$. Hence

$$m'x_1 + m''x_2 = c = \text{const.} \tag{17}$$

The centre of mass S of the triangle moves along the line $y = \beta$; therefore its velocity is $\dot{x}_1$. The kinetic energy of the triangle ABC is equal to $\frac{1}{2}m'\dot{x}_1^2$, and the kinetic energy of the point M is $\frac{1}{2}m''(\dot{x}_2^2 + \dot{y}_2^2)$. Only the weight of the point M does work. Since the weight has the potential $-m''gy_2$, from the principle of conservation of total energy we obtain

$$\tfrac{1}{2}m'\dot{x}_1^2 + \tfrac{1}{2}m''(\dot{x}_2^2 + \dot{y}_2^2) + m''gy_2 = h = \text{const.} \tag{18}$$

The point B has the abscissa $x_1 - \alpha + a$. Denoting by M' the projection of M on the x-axis, we have $\tan\varphi = MM' \,/\, M'B$, from which $\tan\varphi = y_2 \,/\, (x_1 - \alpha + a - x_2)$. Consequently

$$y_2 - (x_1 - x_2 - \alpha + a)\tan\varphi = 0. \tag{19}$$

From equations (17)—(19) we can obtain x_1, x_2, and y_2, as functions of time.

Equations (17) and (19) are equations of the first degree. We can therefore determine from them x_2 and y_2 as linear functions of x_1. We obtain

$$x_2 = Ax_1 + B, \quad y_2 = A'x_1 + B', \tag{20}$$

where A, B and A', B' are certain constants. Differentiating and substituting in (18) we get

$$\tfrac{1}{2}[m' + m''(A^2 + A'^2)]\, \dot{x}_1^2 + m''g(A'x_1 + B') = h.$$

Calculating the derivative, we obtain

$$[m' + m''(A^2 + A'^2)]\, \dot{x}_1\ddot{x}_1 + m''gA'\dot{x}_1 = 0, \tag{21}$$

whence, after dividing by $\dot{x}_1$,

$$\ddot{x}_1 = \text{const}. \tag{22}$$

Therefore the triangle will move with a uniformly accelerated advancing motion.

From equations (20) we obtain, knowing x_1,

$$A'x_2 - Ay_2 = A'B - AB'. \tag{23}$$

Hence the point M will move along a straight line.

In virtue of (20) we have $\ddot{x}_2 = A\ddot{x}_1$ and $\ddot{y}_2 = A'\ddot{x}_1$; therefore according to (22) $\ddot{x}_2 = \text{const}$ and $\ddot{y}_2 = \text{const}$. The projections of the acceleration of the point M are constants; hence the acceleration of the point M is constant. The relative acceleration of the point M with respect to the triangle is also constant, because we obtain it by subtracting the acceleration of the triangle from the acceleration of the point M. The point M will therefore roll down the hypotenuse with a uniformly accelerated motion (relative to the hypotenuse).

Let us also examine whether M does not leave the triangle before reaching the point B.

Let us denote by $\mathbf{R}$ the reaction of the triangle on the point M. Since the weight $\mathbf{Q}$ and the force $\mathbf{R}$ act on the point M, forming projections on the axes x and y, we obtain:

$$m''\ddot{x}_2 = R_x \quad \text{and} \quad m''\ddot{y}_2 = -m''g + R_y.$$

But $\ddot{x}_2 = \text{const}$ and $\ddot{y}_2 = \text{const}$; hence $R_x = \text{const}$ and $R_y = \text{const}$. Consequently $\mathbf{R}$ is constant. The force $\mathbf{R}$ is therefore directed constantly towards the point M which cannot consequently fall away from the triangle ABC before reaching the point B.

§ 3. Rotation about a fixed axis. If a rigid body has a fixed axis l, then it can only rotate about this axis. Let us assume that the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \dots$ act on the body. Let us give the axis l an arbitrary sense and denote by I the moment of inertia of the body with respect to l, by M the moment of the forces with respect to l, and by ω the angular velocity.

The angular momentum with respect to the axis l is $K = I\omega$ ((7), p. 201). According to the theorem about angular momentum with respect to an axis (p. 202) we have $K^{\cdot} = M$; hence

$$I\omega^{\cdot} = M. \tag{1}$$

The angular acceleration is $\varepsilon = \omega^{\cdot}$; therefore

$$I\varepsilon = M. \tag{I}$$

Let Π and Π' be two planes passing through the axis l; let Π be fixed and Π' attached rigidly to the body and rotating together with it. Finally, let φ denote the angle between the planes Π and Π'. Then $\varphi^{\cdot} = \omega$ and $\omega^{\cdot\cdot} = \varepsilon$, whence by (I)

$$I\varphi^{\cdot\cdot} = M. \tag{2}$$

Differential equation (2) has the same form as the equation $mx^{\cdot\cdot} = P$, which defines the motion of a material point along the x-axis.

If the forces $\boldsymbol{P}_i$ or the moment M are given as functions of φ, $\varphi^{\cdot}$, and t (i. e. of the position of the body, the angular velocity, and the time), then equation (2) is a differential equation of the second order, and determines the motion if φ and $\varphi^{\cdot}$ (i. e. the position and angular velocity of the body) are known at the initial moment $t = t_0$.

The kinetic energy of a body rotating about an axis l is $E = \frac{1}{2}I\omega^2$ (p. 363). Let us denote by ω_0 and ω the angular velocities at t_0 and t, and by L_{t_0t} the work of the forces from t_0 to t. Since the forces of reaction holding the axis at rest have their points of application on the axis, they do no work. From the theorem on the equivalence of work and kinetic energy (p. 364) we therefore get

$$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 = L_{t_0t}. \tag{3}$$

The work of the forces is expressed by formula (14), p. 362,

$$L_{t_0t} = \int_{\varphi_0}^{\varphi} M \,\mathrm{d}\varphi, \tag{4}$$

where φ_0 and φ denote the angles of rotation at t_0 and t. If M is a function of φ only, we can obtain L_{t_0t} from formula (4) as a function of the angle φ. Substituting in (3), we obtain a differential equation of the first order.

Example I. Atwood's machine.[1]) At the ends of an inextensible (weightless) string, passing over a perfectly rough material pulley, are hung two heavy points of masses m_1 and m_2. Let r denote the radius of the pulley.

Let us assume that the points move vertically. The paths traversed by the points are equal, and therefore the accelerations of the points are equal in magnitude, but opposite in sense.

Let us denote by p the projection of the acceleration of the point m_1 on the z-axis, directed vertically downwards (*vide* Fig. 132). The pulley is perfectly rough, and consequently the string does not slide along it. Therefore, if the pulley rotates through an angle φ (where we assume $\varphi > 0$, when the point m_1 falls), then the point m_1 will cover a distance $s = r\varphi$. From this $s^{\cdot\cdot} = r\varphi^{\cdot\cdot}$ and hence

$$p = r\varepsilon. \tag{5}$$

Let us denote by $\mathbf{R}_1$ and $\mathbf{R}_2$ the reactions of the string on the points m_1 and m_2, and by R_1 and R_2 their absolute values. The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ are not equal, because the string does not pass over a smooth body.

The reactions of the string and the weights act on the points m_1 and m_2. Consequently:

$$m_1 p = m_1 g - R_1, \quad -m_2 p = m_2 g - R_2. \tag{6}$$

The part of the string from the point m_1 to the pulley acts on the pulley with a force $-\mathbf{R}_1$; similarly the part of the string from the point m_2 to the pulley acts on the pulley with a force $-\mathbf{R}_2$. The moments of these forces with respect to the axis of the pulley are $R_1 r$ and $-R_2 r$, where the moment of the force $-\mathbf{R}_1$ is positive, as the force $-\mathbf{R}_1$ tends to rotate the pulley in the direction assumed previously as positive for the angle φ.

Therefore, denoting by I the moment of inertia of the pulley with respect to its axis, we obtain by (I), p. 374,

$$I\varepsilon = (R_1 - R_2)\, r. \tag{7}$$

From equations (5)—(7) we can determine ε, p, R_1, and R_2. From equations (6) we get $R_1 - R_2 = (m_1 - m_2)\, g - (m_1 + m_2)\, p$. Substituting in formula (7), we obtain $I\varepsilon = (m_1 - m_2)\, rg - (m_1 + m_2)\, rp$, whence by (5)

$$\varepsilon = \frac{(m_1 - m_2)\, rg}{I + (m_1 + m_2)\, r^2}. \tag{8}$$

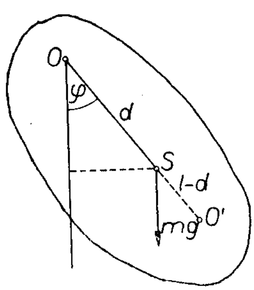

Fig. 288.

Hence we see that $\varepsilon = \text{const}$; consequently in

[1]) cf. p. 193, example 4.

view of (5) $p = \text{const}$. The points m_1 and m_2 will therefore move with a uniformly accelerated motion.

The acceleration p is obtained from (8) and (5). The reactions R_1 and R_2 can be calculated from equations (6).

Compound pendulum. A *compound pendulum* is a rigid body rotating about a horizontal axis under the influence of the force of gravity.

Through the centre of mass S of the pendulum (Fig. 288) let us pass a vertical plane, perpendicular to the axis and cutting it in the point O. Let φ denote the angle between OS and the vertical; the positive sense of rotation is chosen from left to right. Let us finally denote by M the moment of the force of gravity, by I the moment of inertia of the pendulum with respect to the axis of rotation, and let $d = OS$.

The angular acceleration is equal to $\varepsilon = \varphi^{\cdot\cdot}$, and the moment of the force of gravity

$$M = -mgd \sin\varphi$$

(where m denotes the mass of the body), whence by (2), p. 374, $I\varphi^{\cdot\cdot} = -mgd \sin\varphi$, whence

$$\varphi^{\cdot\cdot} = -\frac{mgd}{I} \sin\varphi. \tag{9}$$

Comparing equation (9) with the equation of the simple pendulum ((I), p. 130): $\varphi^{\cdot\cdot} = -\frac{g}{l} \sin\varphi$, we see that if the length l of the simple pendulum satisfies the condition

$$-mgd / I = -g / l, \tag{10}$$

then the motion of the compound pendulum is the same as that of the simple pendulum. From (10) we obtain

$$l = I / md. \tag{11}$$

Therefore: *the motion of a compound pendulum is the same as the motion of a simple pendulum of length* $l = I / md$, where I denotes the moment of inertia with respect to the axis of rotation, m the mass of the pendulum, and d the distance of the centre of mass from the axis of rotation.

Denoting by K the radius of gyration with respect to the axis of rotation, we have $I = mK^2$, whence by (11)

$$l = K^2 / d. \tag{12}$$

The length l is called the *reduced length* of the compound pendulum with respect to the axis of rotation.

Let I_0 denote the moment of inertia, and K_0 the radius of gyration with respect to the axis passing through the centre of gravity and parallel to the axis of rotation. Then (p. 158) $I = I_0 + md^2$, whence $mK^2 = mK_0^2 + md^2$ or $K^2 = K_0^2 + d^2$, and hence by (12)

$$l = \frac{K_0^2}{d} + d. \tag{13}$$

On the line OS let us consider the point O' at a distance l from O. Since $l > d$ by (13), O' will fall beyond the point S. Let us calculate the reduced length l' with respect to the axis of rotation passing through O' and parallel to the axis l passing through O.

Since $O'S = l - d$, we have by (13)

$$l' = \frac{K_0^2}{l-d} + l - d. \tag{14}$$

In view of (13) $l - d = K_0^2 / d$; hence after substitution we get from (14)

$$l' = d + \frac{K_0^2}{d}.$$

Comparing with (13), we see that

$$l' = l.$$

The reduced lengths with respect to the axes passing through O and through O' are consequently equal.

Therefore, if a body is hung on an axis passing through O' and parallel to an axis passing through O, the period of oscillation in both cases is the same (under the same initial angular displacement).

The point O' is called the *centre of oscillation* with respect to the point O.

Determination of the reaction on an axis of rotation. Let us assume that the axis of rotation l is fixed by means of reactions (frictionless) acting on the axis l. Taking an arbitrary point O on the axis l as the centre of reduction, we can replace the reactions by one force $\boldsymbol{R}$ with its origin at O and a force couple of moment $\boldsymbol{H}$. In general $\boldsymbol{R}$ and $\boldsymbol{H}$ change during the motion. To compute $\boldsymbol{R}$ and $\boldsymbol{H}$ we shall use d'Alembert's principle.

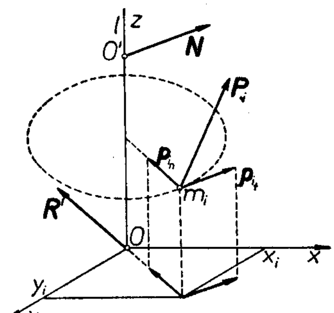

Fig. 289.

Let us select O as the origin of the coordinate system, taking the axis l as the z-axis (Fig. 289). Let us divide the body into small pieces and replace each of them by a material point of equal mass. In this manner we obtain a system of material points $m_1, m_2, \ldots$ having the coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ We shall denote the accelerations of the points of the system by $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots$, and the forces acting on these points by $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$

The forces of inertia $-m_i\boldsymbol{p}_i$ balance the reactions and the forces $\boldsymbol{P}_i$; hence the sum and total moment (with respect to O) are equal to zero. Consequently:

$$\Sigma\boldsymbol{P}_i + \boldsymbol{R} + \Sigma(-m_i\boldsymbol{p}_i) = 0, \tag{15}$$

$$\Sigma\mathrm{Mom}_O\boldsymbol{P}_i + \boldsymbol{H} + \Sigma\mathrm{Mom}_O(-m_i\boldsymbol{p}_i) = 0. \tag{16}$$

Let the body have an angular velocity ω and an angular acceleration ε at the instant t. Let us consider an arbitrary point $m_i(x_i, y_i, z_i)$. Let us resolve the acceleration $\boldsymbol{p}_i$ of this point into a tangential acceleration $\boldsymbol{p}_{i_t}$ and a normal acceleration $\boldsymbol{p}_{i_n}$ (p. 40). We obviously have $\boldsymbol{p}_i = \boldsymbol{p}_{i_t} + \boldsymbol{p}_{i_n}$. The component accelerations are expressed by the formulae $p_{i_t} = r_i\varepsilon$, and $p_{i_n} = r_i\omega^2$ (p. 45), where r_i denotes the distance of the point from the axis of rotation. Consequently:

$$p_{i_{t_x}} = y_i\varepsilon, \qquad p_{i_{t_y}} = -x_i\varepsilon, \qquad p_{i_{t_z}} = 0; \tag{17}$$

$$p_{i_{n_x}} = -x_i\omega^2, \quad p_{i_{n_y}} = -y_i\omega^2, \quad p_{i_{n_z}} = 0. \tag{18}$$

Denoting by m the mass of the body, by x_0, y_0, z_0, the coordinates, and by $\boldsymbol{p}_0$ the acceleration of the centre S of its mass, we obtain from formula (III), p. 195,

$$\Sigma m_i\boldsymbol{p}_i = m\boldsymbol{p}_0. \tag{19}$$

Equation (15) will therefore assume the form

$$\Sigma\boldsymbol{P}_i + \boldsymbol{R} - m\boldsymbol{p}_0 = 0. \tag{20}$$

For the tangential acceleration $\boldsymbol{p}_{0_t}$ and the normal acceleration $\boldsymbol{p}_{0_n}$ of the centre of mass S, we have $\boldsymbol{p}_0 = \boldsymbol{p}_{0_t} + \boldsymbol{p}_{0_n}$. Therefore by (17) and (18) we obtain (putting $i = 0$):

$$p_{0_x} = y_0\varepsilon - x_0\omega^2, \quad p_{0_y} = -x_0\varepsilon - y_0\omega^2, \quad p_{0_z} = 0. \tag{21}$$

Let us now calculate the moments of the forces of inertia. The force of inertia is

$$-m_i\boldsymbol{p}_i = -m_i\boldsymbol{p}_{i_t} - m_i\boldsymbol{p}_{i_n}. \tag{22}$$

Let us denote by $\mathbf{B}$ the moment of the forces of inertia with respect to O, by $\mathbf{B}_t$ the moment of the tangential forces of inertia (i. e. the forces $-m_i \mathbf{p}_{i_t}$), and by $\mathbf{B}_n$ the moment of the normal forces of inertia (i. e. the forces $-m_i \mathbf{p}_{i_n}$). By (22)

$$\mathbf{B} = \mathbf{B}_t + \mathbf{B}_n. \tag{23}$$

The projection of $\mathbf{B}_t$ on the x-axis is (cf. (2), p. 232)

$$B_{t_x} = \Sigma(-m_i p_{i_{t_y}} z_i + m_i p_{i_{t_z}} y_i), \tag{24}$$

whence by (17)

$$\begin{gathered} B_{t_x} = \varepsilon\, \Sigma m_i x_i z_i \text{ and similarly } B_{t_y} = \varepsilon\, \Sigma m_i y_i z_i, \\ B_{t_z} = -\varepsilon\, \Sigma m_i (x_i^2 + y_i^2). \end{gathered} \tag{25}$$

Proceeding in the same way, we obtain:

$$B_{n_x} = \omega^2\, \Sigma m_i y_i z_i, \quad B_{n_y} = \omega^2\, \Sigma m_i x_i z_i, \quad B_{n_z} = 0. \tag{26}$$

By dividing the body into smaller and smaller pieces the sums in formula (24) tend to the products of inertia D_y and D_x, as well as to the moment of inertia I_z with respect to the z-axis (p. 158). In the limit we therefore get from (25) and (26):

$$B_{t_x} = \varepsilon D_y, \quad B_{t_y} = \varepsilon D_x, \qquad B_{t_z} = -\varepsilon I_z, \tag{27}$$

$$B_{n_x} = \omega^2 D_x, \; B_{n_y} = -\omega^2 D_y, \; B_{n_z} = 0, \tag{28}$$

whence, by (23),

$$B_x = \varepsilon D_y + \omega^2 D_x, \quad B_y = \varepsilon D_x - \omega^2 D_y, \quad B_z = -\varepsilon I_z. \tag{29}$$

Forming projections on the coordinate axes, we get from equations (20) and (21):

$$\begin{aligned} &\Sigma P_{i_x} + R_x - m y_0 \varepsilon + m x_0 \omega^2 = 0, \\ &\Sigma P_{i_y} + R_y + m x_0 \varepsilon + m y_0 \omega^2 = 0, \\ &\Sigma P_{i_z} + R_z = 0. \end{aligned} \tag{II}$$

The forces of reaction have their points of application on the axis l; consequently $H_z = 0$. Denoting by $\mathbf{M}$ the moment of the forces $\mathbf{P}_i$ with respect to O, by (16) and (29) we therefore obtain for the projections on the coordinate axes:

$$\begin{aligned} &M_x + H_x + \varepsilon D_y + \omega^2 D_x = 0, \\ &M_y + H_y + \varepsilon D_x - \omega^2 D_y = 0, \\ &M_z - \varepsilon I_z = 0. \end{aligned} \tag{III}$$

The last of the equations (III) was derived previously from the principle of angular momentum (p. 375, formula (I)). From equations (II) and (III) we can calculate **R** and **H**.

Let us now assume that the axis is fixed at two points O and O'. Denote the reactions at these points by **R**′ and **N**, put $d = OO'$, and give the z-axis the direction OO'. We obtain:

$$R_x = R'_x + N_x, \quad R_y = R'_y + N_y, \quad R_z = R'_z + N_z; \tag{30}$$

$$H_x = N_y d, \qquad H_y = -N_x d, \qquad H_z = 0. \tag{31}$$

If we determine **R** and **H** from (II) and (III), then we can calculate from (30) and (31) only the components N_x, N_y, R'_x, R'_y, and the sum $N_z + R'_z$.

Let us assume that O' is in a frictionless bearing (Fig. 204). Then **N** is perpendicular to the axis of rotation; consequently $N_z = 0$.

Therefore, if the point O' is in a frictionless bearing, the reactions can be determined.

Axis of rotation as a central axis of inertia. Under the assumption that the centre of gravity lies on the axis of rotation and that the axis of rotation is one of the central axes of inertia, we have (p. 164):

$$x_0 = 0, \quad y_0 = 0, \quad D_x = 0, \quad D_y = 0.$$

Hence equations (II) and (III) assume the form:

$$\Sigma P_{i_x} + R_x = 0, \; \Sigma P_{i_y} + R_y = 0, \; \Sigma P_{i_z} + R_z = 0; \tag{32}$$

$$M_x + H_x = 0, \quad M_y + H_y = 0, \quad M_z - \varepsilon I_z = 0. \tag{33}$$

We see from this that under this assumption the reactions do not depend on the angular velocity or on the angular acceleration. Therefore they are such as if the body were at rest.

If the forces $\mathbf{P}_1$, $\mathbf{P}_2$, ... are equal to zero, then from equations (32) and (33) we obtain $\mathbf{R} = 0$, $\mathbf{H} = 0$, and $\varepsilon = 0$; hence $\omega =$ const. Since the reactions are equipollent to zero, we can assume that the axis l is not fixed, i. e. that the axis of rotation is free.

Conversely, if we assume that no forces act on the body, and therefore that $\mathbf{P}_i = 0$, $\mathbf{R} = 0$, and $\mathbf{H} = 0$, then equations (II) and (III) assume the form:

$$\begin{aligned} &-y_0\varepsilon + x_0\omega^2 = 0, \quad x_0\varepsilon + y_0\omega^2 = 0, \\ &\varepsilon D_y + \omega^2 D_x = 0, \qquad \varepsilon D_x - \omega^2 D_y = 0, \quad \varepsilon I_z = 0. \end{aligned} \tag{34}$$

From the last of the equations (34) $\varepsilon = 0$; consequently $\omega =$ const. If $\omega \neq 0$ (i. e. when the body is not at rest), we obtain from (34) $x_0 = 0$,

$y_0 = 0$, and $D_x = 0$, $D_y = 0$. The z-axis (i. e. the axis of rotation) is therefore the central axis of inertia.

Hence we have derived the following property of the central axes of inertia:

If a free rigid body on which no forces act rotates about a fixed axis, then this axis is one of the central axes of inertia.

Example 2. A heavy rectangular door $OABC$ (Fig. 290) (where $\overline{OA}$ and $\overline{CB}$ have a vertically upward sense) can rotate about the vertical axis OA which is fixed at the points O_1 and O_2 of the side OA, where $OO_1 = O_2A = d$. At the instant $t = 0$ the door is at rest and a force $\mathbf{P}$ of constant absolute value P, perpendicular to the door and applied constantly at the centre D of the side BC, begins to act. Determine the reactions at the points O_1 and O_2 at the moment t.

Let us denote the mass of the door by m and the moment of inertia with respect to the axis OA by I. Let $OA = a$ and $OC = b$. Let us assume that the force $\mathbf{P}$ revolves the door from right to left with respect to the axis of rotation, which is directed upwards.

The moment of the force $\mathbf{P}$ with respect to the axis of rotation is constant and equal to bP. Consequently $I\varepsilon = bP$, whence $\varepsilon = bP / I$. If the door is homogeneous, then by (8), p. 180, $I = \frac{1}{3}mb^2$. Hence

$$\varepsilon = 3P / mb. \tag{35}$$

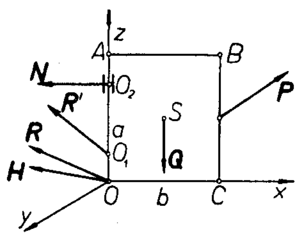

Fig. 290.

The angular acceleration $\varepsilon = \text{const}$; consequently

$$\omega = \varepsilon t = 3Pt / mb. \tag{36}$$

The door will therefore rotate with a uniformly accelerated motion.

We shall now determine the reactions. At the instant t let us take O as the origin of the coordinate system, giving the axes z and x the directions OA and OC. Let us denote by $\mathbf{R}'$ the reaction at O_1, by $\mathbf{N}$ the reaction at O_2, finally by $\mathbf{R}$ the sum, and by $\mathbf{H}$ the moment, of the reactions with respect to O. Assuming that there is a bearing at O_2 (p. 278), we obtain:

$$R_x = R'_x + N_x,\ R_y = R'_y + N_y,\ R_z = R'_z; \tag{37}$$

$$H_x = R'_y d + N_y (a - d),\ H_y = - R'_x d - N_x (a - d),\ H_z = 0. \tag{38}$$

From equations (II) and (III), p. 379, we obtain $\mathbf{R}$ and $\mathbf{H}$, and then we calculate $\mathbf{R}'$ and $\mathbf{N}$ from (37) and (38).

The acting forces are the weight **Q** and the force **P**. The coordinates of the centre of mass S of the door are: $x_0 = \frac{1}{2}b$, $y_0 = 0$, $z_0 = \frac{1}{2}a$, and those of the initial point of the force $\boldsymbol{P}: x = b, y = 0, z = \frac{1}{2}a$. Since the door lies in the zx-plane, $D_x = 0$. The product of inertia D_y is (p. 175)

$$D_y = \int_0^b \int_0^a \varrho xz \, dx \, dz = \int_0^b \tfrac{1}{2}\varrho a^2 x \, dx = \tfrac{1}{4}a^2b^2\varrho,$$

where ϱ denotes the density. Since $ab\varrho = m$, $D_y = \frac{1}{4}mab$. From equations (II) and (III), p. 379, we obtain:

$$R_x + \tfrac{1}{2}mb\omega^2 = 0, \quad -P + R_y + \tfrac{1}{2}mb\varepsilon = 0, \quad -mg + R_z = 0; \quad (39)$$

$$-\tfrac{1}{2}aP + H_x + \tfrac{1}{4}mab\varepsilon = 0, \quad -\tfrac{1}{2}mbg + H_y - \tfrac{1}{4}mab\omega^2 = 0. \quad (40)$$

From equations (39) and (40) we calculate **R** and **H** by means of (35) and (36), and then we determine the reactions **R′** and **N** from equations (37) and (38).

Example 3. A heavy rod AO hangs from a horizontal axis and can only rotate in a vertical plane about its end O. The rod is released freely from a horizontal position. What is the reaction at O when the rod makes an angle φ with the vertical?

Let us denote by **R** the reaction at O, by **Q** the weight of the rod, by m the mass of the body, and by $\boldsymbol{p}_0$ the acceleration of the centre of mass S of the rod (Fig. 291).

From the theorem on the motion of the centre of mass it follows that

$$m\boldsymbol{p}_0 = \boldsymbol{R} + \boldsymbol{Q}. \quad (41)$$

From this equation one can determine **R** if $\boldsymbol{p}_0$ is known. Let us put $l = OS$ and denote by ω and ε the angular velocity and the angular acceleration of the rod, respectively; then the accelerations: tangential p_{0_t} and normal p_{0_n} are:

$$p_{0_t} = l\varepsilon \text{ and } p_{0_n} = l\omega^2.$$

The angular acceleration is obtained from the equation $I\varepsilon = M$. Since the moment of the force of gravity with respect to O is equal to $mgl \sin\varphi$,

$$\varepsilon = mgl \sin\varphi \,/\, I, \quad (42)$$

where I denotes the moment of inertia of the rod with respect to O.

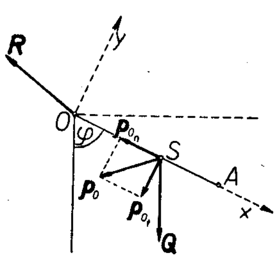

Fig. 291.

The angular velocity ω is calculated by appealing to the principle of equivalence of work and kinetic energy (p. 364). The increase in kinetic energy of the rod is equal to the work of the force of gravity. The kinetic energy of the rod is $E = \frac{1}{2}I\omega^2$, and the work of the weight $L = mgl\cos\varphi$, because the level of the centre of mass was lowered by $h = l\cos\varphi$. Initially the kinetic energy was equal to zero. Therefore $\frac{1}{2}I\omega^2 = mgl\cos\varphi$, whence

$$\omega^2 = 2mgl\cos\varphi \,/\, I. \tag{43}$$

Let us take the direction OS as the positive direction of the x-axis of the coordinate system (x, y). Forming the projections on the axes x and y, we obtain from equations (41):

$$-ml\omega^2 = mg\cos\varphi + R_x, \quad -ml\varepsilon = -mg\sin\varphi + R_y.$$

From (42) and (43) we therefore get:

$$R_x = -mg\cos\varphi[1 + 2ml^2 / I], \; R_y = mg\sin\varphi[1 - ml^2 / I].$$

If the rod is homogeneous, then its length is equal to $a = 2l$, and the moment of inertia $I = \frac{4}{3}ml^2$ (p. 179). Consequently:

$$R_x = -\tfrac{5}{2}mg\cos\varphi, \; R_y = \tfrac{1}{4}mg\sin\varphi,$$

whence

$$|\mathbf{R}| = \tfrac{1}{4}mg\sqrt{1 + 99\cos^2\varphi}.$$

The maximum value of $|\mathbf{R}|$ therefore occurs for $\varphi = 0$ and is

$$|\mathbf{R}| = \tfrac{5}{2}mg.$$

Centre of percussion. Let us assume that the axis of rotation z is a principal axis of inertia at the point O and that the centre of mass lies in the yz-plane (where $y_0 > 0$) at a certain instant t_0(Fig. 292). Consequently:

$$x_0 = 0, \; y_0 > 0, \quad D_x = 0, \; D_y = 0. \tag{44}$$

Equations (II) and (III), p. 379, then assume the form:

$$\begin{gathered} \Sigma P_{i_x} + R_x - my_0\varepsilon = 0, \quad \Sigma P_{i_y} + R_y + my_0\omega^2 = 0, \\ \Sigma P_{i_z} + R_z = 0, \end{gathered} \tag{45}$$

$$M_x + H_x = 0, \quad M_y + H_y = 0, \quad M_z - \varepsilon I_z = 0. \tag{46}$$

Let us assume that the force $\mathbf{P}$, with its point of application at A whose coordinates are x, y, z (Fig. 292), suddenly began to act at the instant t_0. As a result of the action of the force $\mathbf{P}$, the reaction $\mathbf{R}$ and the moment $\mathbf{H}$ changed to $\mathbf{R} + \mathbf{R}'$ and $\mathbf{H} + \mathbf{H}'$; the acceleration ε assumed the value $\varepsilon + \varepsilon'$; the angular velocity, equal to ω at the instant t_0, did not undergo a sudden change. At the instant when the force $\mathbf{P}$ begins to act, equations (II) and (III) assume (in view of (44)) the form:

$$\begin{aligned} &P_x + \Sigma P_{i_x} + R_x + R'_x - my_0(\varepsilon + \varepsilon') = 0, \\ &P_y + \Sigma P_{i_y} + R_y + R'_y + my_0\omega^2 = 0, \\ &P_z + \Sigma P_{i_z} + R_z + R'_z = 0, \end{aligned} \tag{47}$$

$$\begin{aligned} &M'_x + M_x + H_x + H'_x = 0, \\ &M'_y + M_y + H_y + H'_y = 0, \\ &M'_z + M_z - (\varepsilon + \varepsilon')\, I_z = 0, \end{aligned} \tag{48}$$

where $\boldsymbol{M}'$ denotes the moment of the force $\mathbf{P}$ with respect to O. Comparing equations (45) and (46) with (47) and (48), we get:

$$\begin{matrix} P_x + R'_x - my_0\varepsilon' = 0, & P_y + R'_y = 0, & P_z + R'_z = 0, \\ M'_x + H'_x = 0, & M'_y + H'_y = 0, & M'_z - \varepsilon' I_z = 0. \end{matrix} \tag{49}$$

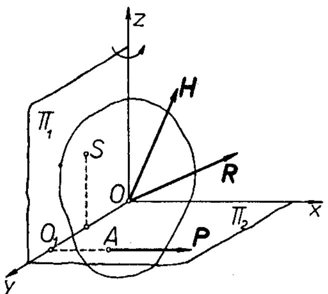

Fig. 292.

From equations (49) we can determine $\mathbf{R}'$ and $\mathbf{H}'$.

Let us assume that $\mathbf{P}$ has the direction of the x-axis and lies in the xy-plane. Then:

$$\begin{matrix} z = 0, & P_y = 0, & P_z = 0, \\ M'_x = 0, & M'_y = 0, & M'_z = P_x y. \end{matrix} \tag{50}$$

From (49) we obtain:

$$\mathbf{H}' = 0, \tag{51}$$

$$\varepsilon' = P_x y \,/\, I_z. \tag{52}$$

From (49), (50) and (52) we get:

$$R'_x = P_x (myy_0 \,/\, I_z - 1), \;\; R'_y = 0, \;\; R'_z = 0. \tag{53}$$

On the y-axis let us consider a point O_1 whose ordinate l is defined by the formula

$$l = I_z \,/\, my_0. \tag{54}$$

If the direction of the force $\mathbf{P}$ passes through O_1, then $y = l$ and hence by (53) $R'_x = 0$, $R'_y = 0$, and $R'_z = 0$, whence

$$\mathbf{R} = 0. \tag{55}$$

The point O_1 is called the *centre of percussion.*

The centre of percussion lies on the line of intersection of the plane Π_1 passing through the axis of rotation and the centre of mass, with the plane Π_2 perpendicular to the axis of rotation at the point O. The centre of percussion lies in Π_1 on the same side of the axis of rotation as the centre of mass. The distance of the centre of percussion from the axis of rotation is defined by formula (54).

If a body is acted upon suddenly by a force whose direction passes through the centre of percussion and is perpendicular to the plane passing through the axis of rotation and the centre of mass, then the reactions supporting the axis of rotation do not change suddenly (the axis does not quiver).

By (53) and (54) we have

$$R'_x = P_x(y \,/\, l - 1). \tag{56}$$

Hence if $y > l$, then $\mathbf{R}'$ and $\mathbf{P}$ have the same senses, and if $y < l$, then $\mathbf{R}$ and $\mathbf{P}$ have opposite senses.

Let us assume that in a compound pendulum a certain plane Π_2, perpendicular to the axis, passes through the centre of mass S and is a central plane. Consequently the axis of rotation is a principal axis of inertia with respect to the point O in which the axis pierces the plane Π_2. The line OS is the intersection of the plane Π_2 with the plane Π_1, passing through the axis of rotation and the centre of mass. The centre of percussion hence lies on the line OS at a distance l from O, defined by formula (54), where obviously $y_0 = OS$. Therefore putting $OS = d$ and $I_z = I$, we get $l = I \,/\, md$. Comparing with formula (11), p. 376, we see that the centre of percussion coincides with the centre of oscillation.

Let us assume that the pendulum is at rest and that the axis of the pendulum lies on two smooth horizontal rods. The plane Π_1 (passing through the axis and the centre of mass) is therefore vertical.

If the pendulum is struck at the centre of percussion in a horizontal direction perpendicular to Π_1, then the axis will not quiver.

If it is struck above the centre of percussion, then by (56) a reaction having a sense opposite to that of striking is necessary to maintain the axis at rest. Since this reaction cannot appear (because the rods on which the axis lies are smooth and cannot therefore induce a horizontal reaction), the axis will move in the direction of striking.

On the other hand, it the pendulum is struck below the centre of percussion, then the axis will move in the direction opposite to that of striking.

§ 4. Plane motion. Plane motion of a plane figure. Let a material figure move in the plane Π and let the acting forces $\mathbf{P}_1, \mathbf{P}_2, \ldots,$ also lie in this plane (Fig. 293). Let us denote by $\mathbf{p}_0$ the acceleration of the centre of mass S of the figure, by m the mass, and by $\mathbf{P}$ the sum of the forces. Then according to (I), p. 364,

$$m\dot{\mathbf{p}}_0 = \mathbf{P}. \qquad \text{(I)}$$

Let us further denote by ω the instantaneous angular velocity and by I_0 the moment of inertia with respect to the centre of mass. The instantaneous motion of the figure can be considered as the composition of an advancing motion with a velocity of the centre of mass and a rotating motion with a velocity ω about the centre of mass. Since the angular momentum of an advancing motion with respect to the centre of mass is zero (p. 200), the angular momentum K of the instantaneous motion with respect to the centre of mass is equal to the angular momentum of the rotating motion. By formula (7), p. 201, we therefore have

$$K = I\omega, \qquad (1)$$

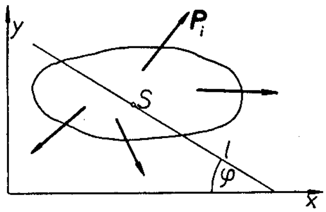

Fig. 293.

whence $K^{\cdot} = I\varepsilon$, where $\varepsilon = \omega^{\cdot}$. Denoting by M the moment of the forces with respect to the centre ot mass, we obtain from (II), p. 364,

$$I\varepsilon = M. \qquad \text{(II)}$$

Equations (I) and (II) define the motion of the material figure in the plane.

Let us consider a fixed line l in the figure, and denote by φ the angle between l and the x-axis, measured clockwise (Fig. 293). The coordinates x_0, y_0, of the centre of mass and the angle φ define the position of the figure.

Forming projections on the coordinate axes, we obtain from equation (I):

$$mx_0^{\cdot\cdot} = P_x, \quad my_0^{\cdot\cdot} = P_y. \qquad \text{(I}')$$

Since $\varphi^{\cdot} = \omega$ and $\varphi^{\cdot\cdot} = \varepsilon$, by (II)

$$I\varphi^{\cdot\cdot} = M. \qquad \text{(II}')$$

From equations (I′) and (II′) we can determine x_0, y_0, and φ.

Plane motion of a body. Let a body move with a plane motion, i. e. let its points move in planes parallel to a certain fixed plane Π, called the directional plane (p. 312). Let us resolve the forces $\{\mathbf{P}_i\}$ acting on the body into the components $\{\mathbf{P}_i'\}$ parallel to Π and into the components $\{\mathbf{P}_i''\}$ perpendicular to Π (Fig. 294). Since the centre of gravity S moves in a plane parallel to Π, its acceleration $\mathbf{p}_0$ lies in the plane Π. By the principle of the motion of the centre of mass we have

$$m\boldsymbol{p}_0 = \Sigma \boldsymbol{P}_i = \Sigma \boldsymbol{P}'_i + \Sigma \boldsymbol{P}''_i,$$

and hence after forming projections on the directional plane

$$m\boldsymbol{p}_0 = \Sigma \boldsymbol{P}'_i. \tag{2}$$

Denoting by l the axis perpendicular to Π and passing constantly through the center of gravity, by I the moment of inertia, and by K the angular momentum with respect to the axis l, we obtain (p. 364) $K^{\cdot} = \Sigma \mathrm{Mom}_l \boldsymbol{P}_i$. But the moment of the forces $\boldsymbol{P}''_i$ with respect to l is zero, because $\boldsymbol{P}''_i \parallel l$; consequently

$$K^{\cdot} = \Sigma \mathrm{Mom}_l P'_i. \tag{3}$$

The instantaneous axis of rotation in a plane motion is perpendicular to the directional plane; the axis l is therefore an instantaneous axis of rotation. Since it passes constantly through the centre of mass, $K = I\omega$ ((7), p. 201), from which $K^{\cdot} = I\omega^{\cdot} = I\varepsilon$ and by (3)

$$I\varepsilon = \Sigma \mathrm{Mom}_l \boldsymbol{P}'_i. \tag{4}$$

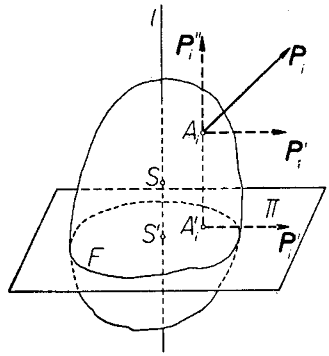

Fig. 294.

In the plane Π let us consider an arbitrary plane figure F attached rigidly to the body: this can be, for example, a section of the body made by the plane Π or the projection of the body on this plane. The motion of the figure F obviously determines the motion of the body. Let us form the projections of the forces $\{\boldsymbol{P}_i\}$ and the centre of gravity S on the plane Π. Equations (2) and (4) define the plane motion of the figure F under the assumption that:

1. the mass of the figure F is equal to the mass of the entire body,

2. the centre of gravity of the figure F is the projection of the centre of mass of the body,

3. the moment of inertia of the figure F with respect to S' is equal to the moment of inertia I of the body with respect to the axis l (Fig. 294).

It follows from this that the plane motion of a body will be determined if we give the projections of the forces on the directional plane, the mass of the body, the projection of its centre of gravity and the moment of inertia of the body with respect to a line passing through the centre of mass and perpendicular to the directional plane.

Example I. A heavy rod AB slides down in a vertical plane with its ends resting on two smooth planes: horizontal and vertical (floor and wall). Hence the forces acting on the rod are: the weight at the centre of the rod and the reactions **N**, **R**, perpendicular to the planes. The components N and R of the reactions with respect to the axes x and y are non-negative (as in Fig. 295).

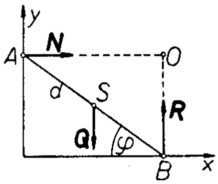

Fig. 295.

Let us denote by x_0, y_0, the coordinates of the centre of mass S of the body, by φ the angle which the rod makes with the coordinate axes, by I_0 the moment of inertia with respect to the centre of mass, and by $2d$ the length of the rod.

From equations (I′) and (II′), p. 386, we obtain:

$$mx_0^{\cdot\cdot} = N, \; my_0^{\cdot\cdot} = -mg + R, \quad I_0\varphi^{\cdot\cdot} = d(N \sin\varphi - R\cos\varphi). \tag{5}$$

In addition to the above equations the following relations hold:

$$x_0 = d\cos\varphi, \quad y_0 = d\sin\varphi. \tag{5'}$$

From equations (5) and (5′) we can obtain a differential equation determining φ as a function of the time t. We shall obtain this equation by applying the principle of equivalence of work and kinetic energy.

The forces **R** and **N** do no work; only the force of gravity does work. Let us note that the velocities of the points A and B have the directions of the axes y and x. Consequently the instantaneous centre of rotation O is the point of intersection of the lines perpendicular to the axes x and y at the points B and A (p. 326). The moment of inertia with respect to O is (cf. (I), p. 159)

$$I = I_0 + md^2, \tag{6}$$

and hence I has a constant value. The kinetic energy E is therefore expressed by the formula $E = \frac{1}{2}I\omega^2 = \frac{1}{2}I\varphi^{\cdot 2}$.

If $\varphi = \varphi_0$ initially, then the work of the force of gravity is $L = mgd(\sin\varphi_0 - \sin\varphi)$. Consequently, under the assumption that $\varphi^{\cdot} = 0$ initially, we obtain

$$\tfrac{1}{2}I\varphi^{\cdot 2} = mgd(\sin\varphi_0 - \sin\varphi). \tag{7}$$

The solution of equation (7) requires a knowledge of the theory of elliptic functions. Nevertheless, we can determine the reactions **N** and **R** without solving the equation if we know φ. With this in view, differentiating equation (7), we obtain $I\varphi^{\cdot}\varphi^{\cdot\cdot} = -mgd\varphi^{\cdot}\cos\varphi$, whence

$$I\varphi^{\cdot\cdot} = -mgd\cos\varphi. \tag{8}$$

By (5′) we have after differentiating:

$$x_0^{\cdot\cdot} = -d\varphi^{\cdot 2}\cos\varphi - d\varphi^{\cdot\cdot}\sin\varphi,\ y_0^{\cdot\cdot} = -d\varphi^{\cdot 2}\sin\varphi + d\varphi^{\cdot\cdot}\cos\varphi. \tag{9}$$

From equations (5) we obtain:

$$N = mx_0^{\cdot\cdot},\ R = my_0^{\cdot\cdot} + mg. \tag{10}$$

From equations (7) and (8) we can determine $\varphi^{\cdot}$ and $\varphi^{\cdot\cdot}$. From equations (9) we then obtain $x_0^{\cdot\cdot}$ and $y_0^{\cdot\cdot}$, whence by (10) we get the reactions **R** and **N**.

Let us calculate the value of the reaction $N = |\mathbf{N}|$. In virtue of (7), (8), and (9),

$$x_0^{\cdot\cdot} = -d\cos\varphi\,\frac{2mgd(\sin\varphi_0 - \sin\varphi)}{I} + d\sin\varphi\,\frac{mgd\cos\varphi}{I};$$

hence by (10)

$$N = mx_0^{\cdot\cdot} = \frac{m^2gd^2\cos\varphi}{I}(3\sin\varphi - 2\sin\varphi_0). \tag{11}$$

Since N must be a non-negative number,

$$3\sin\varphi - 2\sin\varphi_0 \geqq 0. \tag{12}$$

The point A will therefore slide down along the vertical wall as long as the angle φ satisfies the inequality (12). The moment the angle φ reaches the value φ_1 satisfying the equation

$$3\sin\varphi_1 - 2\sin\varphi_0 = 0, \tag{13}$$

the point A will stop sliding along the vertical wall, since the reaction N would then have to become negative, i. e. the wall would have to attract the point A. At that moment, therefore, the rod will fall away from the vertical wall.

Let h_0 denote the initial height of the point A, and h_1 the height of this point at the moment it falls away from the vertical wall. Since $h_0 = 2d\sin\varphi_0$ and $h_1 = 2d\sin\varphi_1$, it follows by (13) that

$$h_1 = \tfrac{2}{3}h_0. \tag{14}$$

Consequently the point A will fall away at $\frac{2}{3}$ of its initial height. After falling away from the wall the motion of the rod will be defined by equations (5) under the assumption that $N = 0$ and $y_0 = d\sin\varphi$.

Example 2. A cylinder of revolution moves down a perfectly rough inclined plane; it will therefore roll. The instantaneous motion of the cylinder will hence be a rotation about a generatrix along which the cylinder is in contact with the plane. Let us assume that this generatrix is horizontal.

The friction does no work because the points of application of the friction (i. e. the points of contact of the cylinder and the plane) have a zero velocity (p. 210). The only force doing work is the weight of the cylinder.

Let us denote by I the moment of inertia of the cylinder with respect to the generatrix, and by ω the angular velocity of rolling at the instant t. The kinetic energy is consequently

$$E = \tfrac{1}{2}I\omega^2. \tag{15}$$

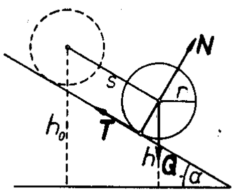

Fig. 296.

Further, let m denote the mass of the cylinder, h the height of the centre of mass at the instant t, and h_0 its height at the instant t_0 (Fig. 296). The work of the force of gravity from t_0 to t is therefore

$$L = mg(h_0 - h). \tag{16}$$

Assuming that the initial angular velocity was ω_0 at the instant t_0, we obtain from (15) and (16) by the principle of the equivalence of work and kinetic energy

$$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 = mg(h_0 - h), \tag{17}$$

from which we can determine ω.

Finally, let s denote the path traversed by the centre of mass from the initial instant t_0 to the instant t. Let us assume that the centre of mass lies on the axis of the cylinder, and let α be the angle made by the inclined plane with the horizontal. Then

$$h_0 - h = s \sin\alpha. \tag{18}$$

Since the velocity of the centre of mass S at the instant t is $v = s^{\cdot} = r\omega$ (where r is the radius of the cylinder), and at the initial instant t_0 it was $v_0 = s_0^{\cdot} = r\omega_0$, by (17) and (18) we obtain

$$Is^{\cdot 2} / 2r^2 - Is_0^{\cdot 2} / 2r^2 = mgs \sin\alpha, \tag{19}$$

whence, differentiating with respect to t, we get $Is^{\cdot}s^{\cdot\cdot} / r^2 = mgs^{\cdot} \sin\alpha$. Consequently

$$p = s^{\cdot\cdot} = mgr^2 \sin\alpha / I. \tag{20}$$

The centre of mass will therefore fall with a uniformly accelerated motion.

The moment of inertia of a solid cylinder (of constant density) with respect to a generatrix is (p. 183, formula (23)) $I = \tfrac{3}{2}mr^2$. Hence by (20)

$p = \frac{2}{3}g \sin\alpha$. The centre of mass will therefore fall with an acceleration smaller than that for a free point, for which $p = g \sin\alpha$ (p. 122).

The moment of inertia of a hollow cylinder (e. g. of a pipe) with respect to the axis is mr^2, and with respect to a generatrix it is $I = 2mr^2$. Hence by (6) $p = \frac{1}{2}g \sin\alpha$. A solid cylinder will therefore fall faster than a hollow one.

If the initial velocity was $\omega_0 = 0$, then the centre of mass traverses a path s in the time

$$t = \sqrt{2s / p} = \sqrt{2sI / mgr^2 \sin\alpha}. \tag{21}$$

Formula (21) can be used to determine the moment of inertia I experimentally.

Let us denote by $\mathbf{T}$ the sum of the frictional forces, by $\mathbf{N}$ the sum of the normal reactions, by $\mathbf{Q}$ the weight, and by $\mathbf{p}$ (as above) the acceleration of the centre of mass of the cylinder. From the theorem on the motion of the centre of mass we have $m\mathbf{p} = \mathbf{T} + \mathbf{N} + \mathbf{Q}$. Since $\mathbf{p}$, $\mathbf{N}$, and $\mathbf{Q}$, are perpendicular to the axis of the cylinder, $\mathbf{T}$ is also perpendicular to the axis of the cylinder. Forming projections on the inclined plane and on the normal to the inclined plane (and putting $T = |\mathbf{T}|$ and $N = |\mathbf{N}|$), we obtain:

$$mp = -T + mg \sin\alpha, \quad 0 = N - mg \cos\alpha,$$

whence by (20):

$$T = mg \sin\alpha(1 - mr^2 / I), \quad N = mg \cos\alpha.$$

Example 3. A circle moves in a vertical plane Π, always remaining tangent to a horizontal line l (Fig. 297). At $t = 0$ the instantaneous motion of the circle was a rotation about the centre of the circle with an angular velocity ω_0. Determine the motion of the circle taking friction into consideration.

Let us take the line l as the x-axis and give to the y-axis a sense vertically upwards. Let us assume that the centre of the circle S is at the same time the centre of mass. Let us denote by r, m, and I, the radius, the mass, and the moment of inertia, of the circle with respect to S, by x_0, y_0, the coordinates of the centre of the circle, by ω and ε the angular velocity and the angular acceleration of the centre S, finally by T and N the components (with respect to the axes x and y) of the friction $\mathbf{T}$ and of the normal reaction $\mathbf{N}$, acting at the point of tangency A. From equations (I) and (II), p. 386, we get:

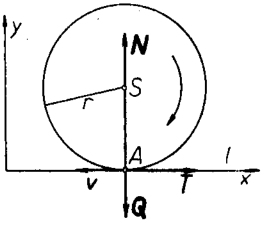

Fig. 297.

$$m\ddot{x}_0 = T, \quad m\ddot{y}_0 = N - mg, \quad I\varepsilon = -Tr. \tag{22}$$

Since $y_0 = r$ constantly, $\ddot{y}_0 = 0$, whence by (22)

$$N = mg. \tag{23}$$

Let v be the component (with respect to the x-axis) of the velocity $\mathbf{v}$ of the point A. Since the instantaneous motion of the circle at the time t is a composition of the advancing motion of the centre of mass with a velocity $\dot{x}_0$ and a rotation about the centre of mass with an angular velocity ω,

$$v = \dot{x}_0 - r\omega, \tag{24}$$

whence by differentiation

$$\dot{v} = \ddot{x}_0 - r\varepsilon. \tag{25}$$

By (22) we have $\ddot{x}_0 = T / m$ and $\varepsilon = - Tr / I$, from which by substitution in equation (25)

$$\dot{v} = (1 / m + r^2 / I)\, T. \tag{26}$$

If $\mathbf{v} \neq 0$, then $\mathbf{T}$ has a sense opposite to that of $\mathbf{v}$ (p. 367); consequently $vT \leqq 0$; whereas if $\mathbf{v} = 0$, then $vT = 0$. Hence we always have $vT \leqq 0$. Therefore, multiplying both sides of the equation (26) by v, we obtain $v\dot{v} = (1 / m + r^2 / I)\, Tv$; consequently

$$v\dot{v} \leqq 0. \tag{27}$$

But $2v\dot{v}$ is the derivative of v^2; hence by (27) the derivative of v^2 is not positive, and consequently v^2 is a non-increasing function. If, therefore, at a certain instant t_1 the value of v reaches 0, then from this instant on $v = 0$ constantly, i. e. from this instant on the circle will roll along the line l.

Let us first examine the motion of the circle from the time $t = 0$ to $t = t_1$. In this interval of time $v \neq 0$; hence $T = \mu N$, where μ denotes the coefficient of friction (p. 367). Taking the sense of the rotation as in the figure, we have $T > 0$; hence according to (23) $T = \mu mg$, whence by substitution in (22)

$$\ddot{x}_0 = \mu g, \quad \varepsilon = -\mu mrg / I. \tag{28}$$

Integrating equations (28) and making use of the conditions $\dot{x}_0 = 0$ and $\omega = \omega_0$ at $t = 0$, we obtain:

$$\dot{x}_0 = \mu gt, \quad \omega = \omega_0 - \mu mgrt / I \tag{29}$$

By (24) we have

$$v = \mu gt(1 + mr^2 / I) - r\omega_0; \tag{30}$$

hence $v = 0$ occurs at the instant

$$t_1 = \frac{r\omega_0}{\mu g(1 + mr^2 / I)}. \tag{31}$$

Since, as we have shown, we shall have $v = 0$ constantly from the moment t_1 on, $v^{\cdot} = 0$, and consequently for $t \geqq t_1$ in virtue of (26) $T = 0$ constantly, or by (22) $x_0 = 0$ and $\varepsilon = 0$.

Therefore for $t \geqq t_1$ the circle will roll with a constant angular velocity ω_1 and the centre of mass will move with a uniform motion with a velocity $v_0 = r\omega_1$.

From formulae (29) and (31) we get

$$\omega_1 = \omega_0 / (1 + mr^2 / I). \tag{32}$$

From the instant t_1 on the kinetic energy is $E_1 = \text{const}$. Since E_1 is equal to the sum of the kinetic energies of the advancing motion with the velocity of the centre of mass $v_0 = r\omega_1$ and of the rotational motion,

$$E_1 = \tfrac{1}{2}mr^2\omega_1^2 + \tfrac{1}{2}I\omega_1^2 = \tfrac{1}{2}I\omega_0^2 / (1 + mr^2 / I). \tag{33}$$

$E_0 = \frac{1}{2}I\omega_0^2$ at $t = 0$; consequently

$$E_1 = E_0 / (1 + mr^2 / I). \tag{34}$$

§ 5. Angular momentum. Let O be an arbitrary point of a moving body. The instantaneous motion of the body is the composition of an instantaneous advancing motion with a velocity $\mathbf{u}$ of the point O and a rotation with an angular velocity $\boldsymbol{\omega}$ about an axis passing through O.

Let us divide the body into small pieces and replace each of them by a material point of the same mass. We shall obtain a system of points $A_1, A_2, \ldots$, of masses $m_1, m_2, \ldots$ At a given moment t let us choose an arbitrary system of coordinates (ξ, η, ζ) whose origin is at O. Let us denote the coordinates of the points $A_1, A_2, \ldots$, by $\xi_1, \eta_1, \zeta_1, \xi_2, \eta_2, \zeta_2, \ldots$

The velocity of the point A_i can be represented in the form (Fig. 298)

$$\mathbf{v}_i = \mathbf{u} + \mathbf{w}_i, \tag{1}$$

where $\mathbf{w}_i$ is the velocity of the instantaneous rotation. By (V), p. 46, we have

$$w_{i\xi} = \eta_i\omega_\zeta - \zeta_i\omega_\eta,\quad w_{i\eta} = \zeta_i\omega_\xi - \xi_i\omega_\zeta,\quad w_{i\zeta} = \xi_i\omega_\eta - \eta_i\omega_\xi. \tag{2}$$

Let $\mathbf{K}_i$ be the moment with respect to O of the momentum $m_i\mathbf{v}_i$ of the point A_i, i.e. $\mathbf{K}_i = \text{Mom}_O(m_i\mathbf{v}_i)$. By (II), p. 18, the projection of K_i on the ξ-axis is $K_{i\xi} = m_i(v_{i\eta}\zeta_i - v_{i\zeta}\eta_i)$, whence by (1) and (2)

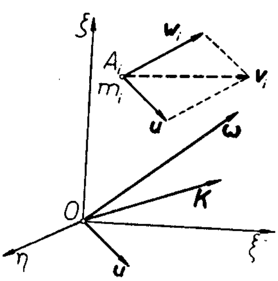

Fig. 298.

$$K_{i\xi} = m_i[(u_\eta + \zeta_i\omega_\xi - \xi_i\omega_\zeta)\,\zeta_i - (u_\zeta + \xi_i\omega_\eta - \eta_i\omega_\xi)\,\eta_i],$$

i. e.

$$K_{i\xi} = \omega_\xi m_i(\zeta_i^2 + \eta_i^2) - \omega_\eta m_i\xi_i\eta_i - \omega_\zeta m_i\xi_i\zeta_i + u_\eta m_i\zeta_i - u_\zeta m_i\eta_i.$$

Since the angular momentum with respect to the point O is $\mathbf{K} = \Sigma\mathbf{K}_i$,

$$K_\xi = \omega_\xi\,\Sigma m_i(\zeta_i^2 + \eta_i^2) - \omega_\eta\,\Sigma m_i\xi_i\eta_i - \omega_\zeta\,\Sigma m_i\xi_i\zeta_i + u_\eta\,\Sigma m_i\zeta_i - u_\zeta\,\Sigma m_i\eta_i.$$

As the body is subdivided into smaller and smaller pieces the sums appearing in the last formula will tend respectively to:

$$I_\xi,\quad D_\zeta,\quad D_\eta,\quad m\zeta_0,\quad m\eta_0,$$

where m denotes the mass of the body, and ξ_0, η_0, ζ_0, the coordinates of the centre of mass. We therefore obtain (after carrying out a similar calculation for the projections K_η and K_ζ):

$$\begin{aligned} K_\xi &= \omega_\xi I_\xi - \omega_\eta D_\zeta - \omega_\zeta D_\eta + m(\zeta_0 u_\eta - \eta_0 u_\zeta),\\ K_\eta &= \omega_\eta I_\eta - \omega_\zeta D_\xi - \omega_\xi D_\zeta + m(\xi_0 u_\zeta - \zeta_0 u_\xi),\\ K_\zeta &= \omega_\zeta I_\zeta - \omega_\xi D_\eta - \omega_\eta D_\xi + m(\eta_0 u_\xi - \xi_0 u_\eta). \end{aligned} \tag{I}$$

Angular momentum with respect to the centre of mass of a body or with respect to its fixed point. If O is the centre of mass, then $\xi_0 = 0$, $\eta_0 = 0$, $\zeta_0 = 0$. On the other hand, if O is fixed, then $u_\xi = 0$, $u_\eta = 0$, $u_\zeta = 0$. In both cases we have by (I):

$$\begin{aligned} K_\xi &= \omega_\xi I_\xi - \omega_\eta D_\zeta - \omega_\zeta D_\eta,\\ K_\eta &= \omega_\eta I_\eta - \omega_\zeta D_\xi - \omega_\xi D_\zeta,\\ K_\zeta &= \omega_\zeta I_\zeta - \omega_\xi D_\eta - \omega_\eta D_\xi. \end{aligned} \tag{II}$$

In particular, if the axes of the coordinate system are principal axes of inertia at the point O, then $D_\xi = 0$, $D_\eta = 0$, $D_\zeta = 0$, and consequently:

$$K_\xi = \omega_\xi I_\xi,\quad K_\eta = \omega_\eta I_\eta,\quad K_\zeta = \omega_\zeta I_\zeta. \tag{III}$$

From formulae (III) it follows that *we can determine the angular momentum if we know the instantaneous angular velocity and conversely.*

The directions of the angular momentum and the angular velocity are in general different. The scalar product $\mathbf{K}\cdot\boldsymbol{\omega}$ is by (III)

$$\mathbf{K}\cdot\boldsymbol{\omega} = K_\xi\omega_\xi + K_\eta\omega_\eta + K_\zeta\omega_\zeta = \omega_\xi^2 I_\xi + \omega_\eta^2 I_\eta + \omega_\zeta^2 I_\zeta;$$

consequently if $\boldsymbol{\omega} \neq 0$, then $\mathbf{K}\boldsymbol{\omega} > 0$.

Therefore: *the angular momentum forms an acute angle with the angular velocity vector.*

We shall now prove the following

Theorem. *If the angular momentum* **K** *or the angular velocity vector* $\boldsymbol{\omega}$ *have the direction of one of the principal axes of inertia, then the angular momentum and the angular velocity have the same direction and conversely.*

Proof. Let us take as the ξ-axis that principal axis of inertia whose direction is the direction of the angular momentum **K**. Then $K_\eta = 0$ and $K_\zeta = 0$, whence by (III) $\omega_\eta = 0$ and $\omega_\zeta = 0$. Therefore the vector $\boldsymbol{\omega}$ has the direction of the ξ-axis, i. e. of the angular momentum.

The proof is carried out in a similar manner if $\boldsymbol{\omega}$ has the direction of one of the principal axes of inertia.

Conversely, if **K** and $\boldsymbol{\omega}$ have the same direction, we take this direction as the direction of the ξ-axis. Then $\omega_\xi = 0$ and $\omega_\zeta = 0$, as well as $K_\eta = 0$ and $K_\zeta = 0$, whence by (II) $K_\xi = I_\xi \omega_\xi$, $0 = -\omega_\xi D_\zeta$, and $0 = -\omega_\xi D_\eta$, and hence $D_\zeta = 0$, $D_\eta = 0$. The ξ-axis is therefore a principal axis of inertia, q. e. d.

If the point O is a spherical point, i. e. if $I_\xi = I_\eta = I_\zeta$, then, denoting the moments of inertia by I, we have by (III) $K_\xi = I\omega_\xi$, $K_\eta = I\omega_\eta$, $K_\zeta = I\omega_\zeta$, whence

$$\mathbf{K} = I\boldsymbol{\omega}. \tag{3}$$

Therefore: *if the centre of mass (or a fixed point of a body) is a spherical point, then the angular momentum constantly has the direction and sense of the angular velocity.*

Derivative of the angular momentum. Let **K** be the angular momentum of a body with respect to an arbitrary point O of this body and let (x, y, z) be a fixed system of coordinates with its origin at O', and (ξ, η, ζ) an arbitrary moving system of coordinates with its origin at O (Fig. 299). Let us denote by $\mathbf{u}$ the velocity of the point O, and by $\boldsymbol{\omega}'$ the instantaneous angular velocity of the system (ξ, η, ζ). Let us draw a vector $\overline{OA} = \mathbf{K}$ from the point O. Putting $\overline{O'O} = \mathbf{r}$ and $\overline{O'A} = \boldsymbol{\rho}$, we obtain $\boldsymbol{\rho} = \mathbf{r} + \mathbf{K}$, whence $\mathbf{K} = \boldsymbol{\rho} - \mathbf{r}$. Calculating the derivative, we obtain $\mathbf{K}^\cdot = \boldsymbol{\rho}^\cdot - \mathbf{r}^\cdot$. But $\mathbf{r}^\cdot = \mathbf{u}$, and $\boldsymbol{\rho}^\cdot$ is equal to the absolute velocity $\mathbf{v}_a$ of the point A with respect to the fixed system $O'(x, y, z)$. Consequently

$$\mathbf{K}^\cdot = \mathbf{v}_a - \mathbf{u}. \tag{4}$$

Let $\mathbf{v}_r$ be the relative velocity of the point A with respect to the system (ξ, η, ζ), and $\mathbf{v}_t$ the velocity of transport. Hence (p. 57)

$$\mathbf{v}_a = \mathbf{v}_r + \mathbf{v}_t, \tag{5}$$

whence by (4)

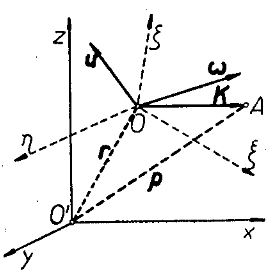

Fig. 299.

$$\mathbf{K}^{\cdot} = \mathbf{v}_r + (\mathbf{v}_t - \mathbf{u}). \tag{6}$$

In the system (ξ, η, ζ) the point A has the coordinates K_ξ, K_η, K_ζ. Consequently

$$v_{r_\xi} = K^{\cdot}_\xi,\ v_{r_\eta} = K^{\cdot}_\eta,\ v_{r_\zeta} = K^{\cdot}_\zeta. \tag{7}$$

The instantaneous motion of the system (ξ, η, ζ) is the composition of an advancing motion with a velocity $\mathbf{u}$ of the point O and a rotation with an instantaneous angular velocity $\boldsymbol{\omega}'$ about an axis passing through O. Hence (p. 62)

$$\mathbf{v}_t = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega}' = \mathbf{u} + \mathbf{K} \times \boldsymbol{\omega}'. \tag{8}$$

Taking

$$\mathbf{w} = \mathbf{K} \times \boldsymbol{\omega}' \tag{9}$$

we therefore obtain by (8) and (6)

$$\mathbf{K}^{\cdot} = \mathbf{v}_r + \mathbf{w}. \tag{10}$$

From (9) we get:

$$w_\xi = K_\eta\omega'_\zeta - K_\zeta\omega'_\eta,\ w_\eta = K_\zeta\omega'_\xi - K_\xi\omega_\zeta,\ w_\zeta = K_\xi\omega'_\eta - K_\eta\omega'_\xi. \tag{11}$$

Let us denote the projections of the vector $\mathbf{K}^{\cdot}$ on the axes ξ, η, and ζ, by $(\mathbf{K}^{\cdot})_\xi$, $(\mathbf{K}^{\cdot})_\eta$, and $(\mathbf{K}^{\cdot})_\zeta$. From (10) in virtue of (7) and (11) we obtain:

$$\begin{gathered}(\mathbf{K}^{\cdot})_\xi = K^{\cdot}_\xi + K_\eta\omega'_\zeta - K_\zeta\omega'_\eta,\ (\mathbf{K}^{\cdot})_\eta = K^{\cdot}_\eta + K_\zeta\omega'_\xi - K_\xi\omega'_\zeta,\\ (\mathbf{K}^{\cdot})_\zeta = K^{\cdot}_\zeta + K_\xi\omega'_\eta - K_\eta\omega'_\xi.\end{gathered} \tag{IV}$$

Formulae (IV) determine the projections of the derivative of the angular momentum $\mathbf{K}^{\cdot}$ with respect to O on the axes ξ, η, ζ of the moving system in terms 1° of the projections K_ξ, K_η, K_ζ, of the angular momentum $\mathbf{K}$ on these axes, 2° of the derivatives $K^{\cdot}_\xi$, $K^{\cdot}_\eta$, $K^{\cdot}_\zeta$, of the projections K_ξ, K_η, K_ζ, and 3° of the projections ω'_ξ, ω'_η, ω'_ζ, of the instantaneous velocity of the system (ξ, η, ζ) — but not of the body! — on the axes of this system.

One should note the difference between the symbols $(\mathbf{K}^{\cdot})_\xi$ and $K^{\cdot}_\xi$. The value of the first symbol is obtained by first calculating the derivative and then forming the projection on the ξ-axis; whereas the value of the second symbol is obtained, conversely, by projecting first the vector $\mathbf{K}$ on the ξ-axis and then calculating the derivative of the projection. As formula (IV) indicates, in general $(\mathbf{K}^{\cdot})_\xi \neq K^{\cdot}_\xi$.

Let us assume that O is a fixed point or the centre of mass and that the axes ξ, η, ζ constantly have the directions of the principal axes of inertia of the body at the point O. In this case the instantaneous angular velocity $\boldsymbol{\omega}$ of the body is equal to the instantaneous angular velocity of the coordinate system (ξ, η, ζ):

$$\boldsymbol{\omega} = \boldsymbol{\omega}'. \tag{12}$$

Since by (III), p. 394,

$$K_\xi = I_\xi\omega_\xi, \quad K_\eta = I_\eta\omega_\eta, \quad K_\zeta = I_\zeta\omega_\zeta, \tag{13}$$

as I_ξ, I_η, I_ζ, are fixed, we get:

$$K_\xi^{\cdot} = I_\xi\omega_\xi^{\cdot}, \quad K_\eta^{\cdot} = I_\eta\omega_\eta^{\cdot}, \quad K_\zeta^{\cdot} = I_\zeta\omega_\zeta^{\cdot}. \tag{14}$$

Substituting the values from (12)—(14) in (IV), we obtain:

$$\begin{gathered}(\mathbf{K}^{\cdot})_\xi = I_\xi\omega_\xi^{\cdot} + (I_\eta - I_\zeta)\,\omega_\eta\omega_\zeta, \ (\mathbf{K}^{\cdot})_\eta = I_\eta\omega_\eta^{\cdot} + (I_\zeta - I_\xi)\,\omega_\zeta\omega_\xi, \\ (\mathbf{K}^{\cdot})_\zeta = I_\zeta\omega_\zeta^{\cdot} + (I_\xi - I_\eta)\,\omega_\xi\omega_\eta.\end{gathered} \tag{V}$$

Formulae (V) refer to a system of coordinates whose origin is the centre of mass or a fixed point of a body and whose axes constantly have the directions of the principal axes of inertia.

§ 6. Euler's equations. We shall now consider the motion a body acted on by forces executes if it has one fixed point O, and is therefore only capable of rotating about this point. For it is to this case that we can reduce the investigation of the motion of a rigid body under the influence of forces in the most general case.

Let $\mathbf{K}$ be the angular momentum and $\mathbf{M}$ the total moment of the forces with respect to O. Then according to the principle of angular momentum (II), p. 364,

$$\mathbf{K}^{\cdot} = \mathbf{M}. \tag{1}$$

Let us note that $\mathbf{M}$ does not depend on a force applied at O, because its moment with respect to O is zero.

Let us choose two coordinate systems having the origin O: a fixed system (x, y, z) and a moving system (ξ, η, ζ) whose axes are the principal axes of inertia with respect to O. Let A, B, C, denote the moments of inertia of the body with respect to the principal axes of inertia (i. e. to the axes ξ, η, ζ), and let $\boldsymbol{\omega}$ denote the instantaneous angular velocity of the body. By (1) and (V) we get:

$$\begin{aligned}A\omega_\xi^{\cdot} + (B - C)\,\omega_\eta\omega_\xi &= M_\xi, \\ B\omega_\eta^{\cdot} + (C - A)\,\omega_\zeta\omega_\xi &= M_\eta, \\ C\omega_\zeta^{\cdot} + (A - B)\,\omega_\xi\omega_\eta &= M_\zeta.\end{aligned} \tag{I}$$

Equations (I) are called *Euler's equations.*

Equations (I) serve to determine ω_ξ, ω_η, ω_ζ, as functions of the time t. Knowing ω_ξ, ω_η, ω_ζ, we can define the position of the moving system (ξ, η, ζ) and hence also the position of the body by means of Euler's angles ϑ, φ, ψ (p. 354), calculated from the differential equations (II), p. 356. In this manner, by means of Euler's equations (I) and equations

(II), p. 356, we can determine the motion of the body. The solution of these equations presents many difficulties and not always can it be carried out. However, we shall meet with some cases in which these solutions can be obtained. The most important of these is the case when no forces except the reaction at the point O act on the body.

If we know the motion of the body, then we can calculate the reaction $\boldsymbol{R}$ applied at O. For let us denote by $\boldsymbol{P}$ the sum of the acting forces, by m the mass of the body, and by $\boldsymbol{p}_0$ the acceleration of the centre of mass. By the theorem on the motion of the centre of mass ((I), p. 364), we hence have $m\boldsymbol{p}_0 = \boldsymbol{P} + \boldsymbol{R}$, whence

$$\boldsymbol{R} = m\boldsymbol{p}_0 - \boldsymbol{P}. \tag{2}$$

Remark 1. Euler's equations (I) also hold when the point O is not a fixed point, but the centre of mass of the body, for then the theorem concerning the angular momentum $\boldsymbol{K}^{\cdot} = \boldsymbol{M}$ (p. 364) holds, and the formulae (II), p. 356, are true for any point O.

Remark 2. Making use of formulae (IV), p. 396, we can give equations which are more general than Euler's equations.

Let O be a fixed point or the centre of mass, and (ξ, η, ζ) an arbitrary system of coordinates with origin at O and having an instantaneous angular velocity $\boldsymbol{\omega}'$. Since $\boldsymbol{K}^{\cdot} = \boldsymbol{M}$, by formulae (IV), p. 396, we obtain:

$$\begin{aligned} K_\xi^{\cdot} + K_\eta\,\omega_\zeta' - K_\zeta\,\omega_\eta' &= M_\xi, \\ K_\eta^{\cdot} + K_\zeta\,\omega_\xi' - K_\xi\,\omega_\zeta' &= M_\eta, \\ K_\zeta^{\cdot} + K_\xi\,\omega_\eta' - K_\eta\,\omega_\xi' &= M_\zeta. \end{aligned} \tag{II}$$

Motion of an unconstrained rigid body. Let us take the centre of mass S of a body as the origin of the coordinate system (x, y, z), moving with an advancing motion relative to an inertial frame. The motion of the body in space will be defined if we determine the motion of the centre of mass S and the motion of the body relative to S, i. e. relative to the system (x, y, z).

The motion of the centre of mass can be obtained from equations (I), p. 364.

On the other hand, in order to determine the motion of the body relative to the system (x, y, z), we can assume that this system is at rest (p. 135) and that in addition to the forces acting on the body, only the forces of transport act on it (because the forces of Coriolis are zero (p. 136)). The acceleration of transport is equal to the acceleration $\boldsymbol{p}_0$ of the centre of mass and is common to all the points of the body (p. 60). If we consider the body as a system of material points $m_1, m_2, \ldots$, then the forces of transport are $-m_1\boldsymbol{p}_0, -m_2\boldsymbol{p}_0, \ldots$ The forces of transport are therefore proportional to the masses and have the same directions as well as senses. Consequently (p. 239) the forces of transport have a resultant $\mathbf{R}$ whose origin is at the centre of mass:

$$\mathbf{R} = -m_1\mathbf{p}_0 - m_2\mathbf{p}_0 - \ldots = -m\mathbf{p}_0,$$

where m denotes the mass of the body. Denoting the sum of the acting forces by $\mathbf{P}$, we obtain from the theorem on the motion of the centre of mass $m\mathbf{p}_0 = \mathbf{P}$, whence $\mathbf{R} = -\mathbf{P}$.

Since the centre of mass S is fixed relative to the system (x, y, z), and the force of transport has its origin at S, *the motion of the body relative to the centre of mass* (and consequently also relative to the system (x, y, z)) *is such as if the centre of mass were fixed and the body were acted upon by the same forces.*

The motion of a body relative to the centre of mass is therefore independent of the motion of the centre of mass itself and we can determine it by means of Euler's equations.

We see from this that the investigation of the motion of a body in the most general case does indeed reduce to the investigation of the motion of the centre of mass and the rotation of the body about a fixed point.

§ 7. Rotation of a body about a point under the action of no forces. Let us assume that no forces act on a rigid body having a fixed point O. In this case the moment of the forces is $\mathbf{M} = 0$; hence Euler's equations (I), p. 397, assume the form:

$$A\dot{\omega}_\xi + (B - C)\,\omega_\eta\omega_\zeta = 0, \quad B\dot{\omega}_\eta + (C - A)\,\omega_\zeta\omega_\xi = 0, \quad C\dot{\omega}_\zeta + (A - B)\,\omega_\xi\omega_\eta = 0. \tag{I'}$$

Equations (I′) also hold under the assumption alone that $\mathbf{M} = 0$ constantly, i. e. that the forces acting on the body have a resultant whose direction constantly passes through the point O. It follows from this that equations (I′) also apply to the motion of a heavy rigid body having a fixed centre of gravity when no forces other than gravity act on the body.

The solution of equations (I′) in the general case requires a knowledge of the theory of elliptic functions. Here we shall give the solutions of those equations only in the cases when the ellipsoid of inertia with respect to O is a sphere or an ellipsoid of revolution, i. e. when all three or at least two of the numbers A, B, C, are equal.

At present we shall deduce certain general propositions from equations (I′).

Angular momentum and kinetic energy. Since $\mathbf{M} = 0$, from the theorem concerning angular momentum it follows that *the angular momentum* $\mathbf{K}$ *is a constant vector.*

Since $|\mathbf{K}|^2 = K_\xi^2 + K_\eta^2 + K_\zeta^2$, by (III), p. 394, we get, putting $I_\xi = A$, $I_\eta = B$, and $I_\zeta = C$,

$$|\mathbf{K}|^2 = A^2\omega_\xi^2 + B^2\omega_\eta^2 + C^2\omega_\zeta^2 = \text{const.} \tag{1}$$

Let us multiply both sides of Euler's equations by $\omega_\xi, \omega_\eta, \omega_\zeta$, and add. We obtain $A\dot{\omega}_\xi\omega_\xi + B\dot{\omega}_\eta\omega_\eta + C\dot{\omega}_\zeta\omega_\zeta = 0$, which can be written in the form $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}(A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2) = 0$, whence

$$A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2 = \text{const.} \tag{2}$$

In order to give equation (2) a meaning, let us consider the angles α, β, γ, which $\boldsymbol{\omega}$ makes with the axes ξ, η, ζ. We therefore have $\omega_\xi = |\boldsymbol{\omega}| \cos\alpha$, $\omega_\eta = |\boldsymbol{\omega}| \cos\beta$, $\omega_\zeta = |\boldsymbol{\omega}| \cos\gamma$, whence

$$A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2 = (A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma)\,|\boldsymbol{\omega}|^2. \tag{3}$$

Let I be the moment of inertia of a body with respect to the instantaneous axis of rotation. In virtue of formula (I), p. 162, $I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma$, and consequently by (3) the left side of (2) is equal to $I|\boldsymbol{\omega}|^2$. Now, since the kinetic energy of the body is $E = \frac{1}{2}I|\boldsymbol{\omega}|^2$ (p. 364),

$$2E = A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2 = \text{const.} \tag{4}$$

From this it is apparent, that equation (2) expresses the fact that the *kinetic energy of the body is constant.*

By (III), p. 394, we further have $\mathbf{K}\boldsymbol{\omega} = A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2$, whence, by (4), $\mathbf{K}\boldsymbol{\omega} = \text{const}$. Since $\mathbf{K}\boldsymbol{\omega} = |\mathbf{K}| \operatorname{Proj}_{\mathbf{K}}\boldsymbol{\omega}$ and according to (1) $|\mathbf{K}| = \text{const}$, $\operatorname{Proj}_{\mathbf{K}}\boldsymbol{\omega} = \text{const}$.

Therefore: *the projection of the instantaneous angular velocity on the direction of the angular momentum is constant.*

Let us suppose that the direction of the angular momentum at $t = 0$ was the same as the direction of the instantaneous angular velocity. Therefore $\mathbf{K}$ and $\boldsymbol{\omega}$ had (p. 394) the direction of one of the principal axes of inertia, e. g. the ζ-axis. Consequently at $t = 0$:

$$\omega_\xi = 0, \quad \omega_\eta = 0 \quad \text{and} \quad \omega_\zeta = \omega_\zeta^0, \tag{5}$$

where ω_ζ^0 denotes the projection of $\boldsymbol{\omega}$ on the ζ-axis at $t = 0$.

Euler's equations (I′) are differential equations of the first order. From the theory of differential equations it is known that there exists only one solution satisfying the conditions (5) at $t = 0$. This unique solution is

$$\omega_\xi = \text{const} = 0, \quad \omega_\eta = \text{const} = 0, \quad \omega_\zeta = \text{const} = \omega_\zeta^0,$$

because it satisfies the conditions (5) for $t = 0$, and, as it is easily verified, Euler's equations (I'). The vector $\boldsymbol{\omega}$ therefore has a constant magnitude and it always has the direction of the principal axis of inertia ζ; consequently (p. 394) $\boldsymbol{\omega}$ likewise has the direction of the angular momentum $\mathbf{K}$. And, since the angular momentum $\mathbf{K}$ maintains a constant direction in space, the direction of the vector $\boldsymbol{\omega}$ is also constant.

Therefore: *if the instantaneous angular velocity initially has the direction of a principal axis of inertia, then* (under the assumption that the moment of the forces with respect to a fixed point is zero) *the motion of the body is a rotation about a fixed axis with a constant angular velocity.*

Rotation about a spherical point. Let us assume that the point O is a spherical point, i. e. that $A = B = C$. Euler's equations (I') then assume the form $\dot{\omega}_\xi = 0$, $\dot{\omega}_\eta = 0$, $\dot{\omega}_\zeta = 0$, i. e.

$$\omega_\xi = c_1, \quad \omega_\eta = c_2, \quad \omega_\zeta = c_3. \tag{II'}$$

It follows from this that the angular velocity is constant in magnitude. The point O is by hypothesis a spherical point; therefore by (3), p. 395 (putting $I = A$), we have $\mathbf{K} = A\boldsymbol{\omega}$, and consequently the instantaneous axis of rotation has the direction of the angular momentum; and because $\mathbf{K}$ is a constant vector, the instantaneous axis of rotation has a constant direction in space. In view of this the body rotates about a fixed axis with a constant angular velocity.

Therefore: *if the point O is a spherical point* (i. e. if $A = B = C$), *then the motion of a body under the action of forces whose moment with respect to O is zero, is a rotation about a fixed axis with a constant angular velocity.*

Rotation about a point whose ellipsoid of inertia is an ellipsoid of revolution. Let us assume that $A = B$, i. e. that the ellipsoid of inertia with respect to the point is one of revolution. This case occurs if the body possesses e. g. an axis of symmetry passing through O. Euler's equations (I') then assume the form:

$$\dot{\omega}_\xi + \frac{A - C}{A}\omega_\zeta\omega_\eta = 0, \quad \dot{\omega}_\eta - \frac{A - C}{A}\omega_\zeta\omega_\xi = 0, \quad \dot{\omega}_\zeta = 0. \tag{II''}$$

The third of the equations (II'') gives

$$\omega_\zeta = c = \text{const}. \tag{6}$$

From equation (4), putting $A = B$, we obtain

$$A(\omega_\xi^2 + \omega_\eta^2) + C\omega_\zeta^2 = \text{const};$$

hence in virtue of (6) we have

$$\omega_\xi^2 + \omega_\eta^2 = c_1^2 = \text{const}. \tag{7}$$

Since $|\boldsymbol{\omega}|^2 = \omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2$, by (6) and (7)

$$|\boldsymbol{\omega}|^2 = c^2 + c_1^2 = \text{const}. \tag{8}$$

Hence: *the instantaneous angular velocity is constant in magnitude.* Let us set

$$\frac{A - C}{A}\,\omega_\zeta = h. \tag{9}$$

Since $\omega_\zeta = \text{const}$ in view of (6), $h = \text{const}$ and the first two equations (II′) assume the form:

$$\dot{\omega}_\xi + h\omega_\eta = 0, \quad \dot{\omega}_\eta - h\omega_\xi = 0. \tag{10}$$

Let $h \neq 0$. Differentiating the first of the equations (10), we get $\ddot{\omega}_\xi + h\dot{\omega}_\eta = 0$ or $\dot{\omega}_\eta = -\ddot{\omega}_\xi / h$, whence by substituting in the second of the equations (10) we obtain after multiplying by h

$$\ddot{\omega}_\xi + h^2\omega_\xi = 0. \tag{11}$$

The general solution of equation (11) has the form

$$\omega_\xi = a \sin ht + b \cos ht, \tag{12}$$

where a and b are arbitrary constants. The first of the equations (10) gives $\omega_\eta = -\dot{\omega}_\xi / h$, whence by (12)

$$\omega_\eta = -a \cos ht + b \sin ht. \tag{13}$$

Equations (6), (12), and (13), represent the general solution of Euler's equations (II″) also when $h = 0$. The solution contains three arbitrary constants a, b, c, which are determined from the initial conditions.

Determination of Euler's angles. We shall now consider the determination of Euler's angles by means of equations (6), (12), and (13), and equations (II), p. 356.

Since the angular momentum $\mathbf{K}$ is a constant vector, we can take the direction of the angular momentum as the direction of the z-axis. The projection of the angular momentum on the ζ-axis is $K_\zeta = |\mathbf{K}| \cos\vartheta$. On the other hand (putting $I_\zeta = C$) we have by (III), p. 394, $K_\zeta = C\omega_\zeta$; therefore $|\mathbf{K}| \cos\vartheta = C\omega_\zeta$, whence

$$\cos\vartheta = C\omega_\zeta / |\mathbf{K}|. \tag{14}$$

Since $|\mathbf{K}| = \text{const}$ and $\omega_\zeta = \text{const}$,

$$\vartheta = \vartheta_0 = \text{const}. \tag{15}$$

If $\vartheta_0 = 0$ or $\vartheta_0 = \pi$, then the angular momentum $\mathbf{K}$ constantly has the direction of the axis of inertia ζ; consequently according to the theorem given on p. 395 the instantaneous angular velocity has the direction of the angular momentum. Similarly, if $\vartheta_0 = \frac{1}{2}\pi$, then by (14) $\omega_\zeta = 0$; hence ((III), p. 394) $K_\xi = A\omega_\xi$, $K_\eta = A\omega_\eta$, $K_\zeta = 0$, whence $\mathbf{K} = A\boldsymbol{\omega}$; therefore the angular momentum has the direction of the instantaneous angular velocity. From the theorem given on p. 395 we conclude, therefore, that *if $\vartheta_0 = 0$ or π or $\frac{1}{2}\pi$, then the motion of a body is a rotation about a fixed axis with a constant angular velocity.*

Let us now assume that $\vartheta_0 \neq 0$, $\vartheta_0 \neq \pi$, and $\vartheta_0 \neq \frac{1}{2}\pi$. Since $\vartheta = \vartheta_0 = \text{const}$, the ζ-axis describes a cone of revolution whose axis is the z-axis. Substituting the values ω_ξ and ω_η from equations (12) and (13) in equations (II), p. 356, we obtain (because $\vartheta^\cdot = 0$):

$$a \sin(ht - \varphi) + b \cos(ht - \varphi) = 0, \tag{16}$$

$$\psi^\cdot = [a \cos(ht - \varphi) - b \sin(ht - \varphi)] / \sin\vartheta_0, \tag{17}$$

$$\varphi^\cdot = \omega_\zeta - [a \cos(ht - \varphi) - b \sin(ht - \varphi)] \cos\vartheta_0. \tag{18}$$

Were $a = 0$ and $b = 0$, then by (12) and (13) we should have $\omega_\xi = 0$ and $\omega_\eta = 0$; hence $\boldsymbol{\omega}$ would have the direction of the axis of inertia ζ, and consequently of the angular momentum $\mathbf{K}$ (in virtue of the theorem on p. 395). The ζ-axis would therefore have the direction of the z-axis and ϑ_0 would be zero or π, contrary to hypothesis. One of the numbers a and b is therefore different from zero, whence by (16) $ht - \varphi = \text{const}$.

Let $\varphi = \varphi_0$ for $t = 0$. Consequently $ht - \varphi = -\varphi_0$, i. e.

$$\varphi = ht + \varphi_0. \tag{19}$$

Substituting this value of φ in (17) and (18), we get:

$$\psi^\cdot = (a \cos\varphi_0 + b \sin\varphi_0) / \sin\vartheta_0, \tag{20}$$

$$h = \omega_\zeta - (a \cos\varphi_0 + b \sin\varphi_0) \cot\vartheta_0. \tag{21}$$

Since $\vartheta_0 \neq 0$, $\vartheta_0 \neq \pi$, and $\vartheta_0 \neq \frac{1}{2}\pi$, from (21) we obtain $a \cos\varphi_0 + b \sin\varphi_0 = (\omega_\zeta - h) \tan\vartheta_0$, whence by (20)

$$\psi^\cdot = (\omega_\zeta - h) / \cos\vartheta_0. \tag{22}$$

Substituting in equations (19) and (22) the value of h from equation (9), we obtain together with equation (15):

$$\varphi^\cdot = \frac{A - C}{A}\omega_\zeta, \quad \psi^\cdot = \frac{C}{A \cos\vartheta_0}\omega_\zeta, \quad \vartheta^\cdot = 0. \tag{23}$$

Integrating equations (23) and assuming that $\varphi = \varphi_0$, $\psi = \psi_0$, and $\vartheta = \vartheta_0$, for $t = 0$, we get:

$$\varphi = \frac{A - C}{A} \omega_\zeta t + \varphi_0, \quad \psi = \frac{C}{A \cos\vartheta_0} \omega_\zeta t + \psi_0, \quad \vartheta = \vartheta_0. \qquad (24)$$

Since according to (6) $\omega_\zeta = \text{const}$, from (23) it follows that $\varphi^{\cdot} = \text{const}$ and $\psi^{\cdot} = \text{const}$.

Consequently: *the motion of a body is the composition of two rotations, one of which is about the fixed axis* ζ *in the body, and the other about the fixed axis* z *in space. The angular velocity of both rotations is constant.*

Such a motion was called a *steady precession* (p. 356). The relation between the two angular velocities is according to (23)

$$\varphi^{\cdot} / \psi^{\cdot} = (A - C) \cos\vartheta_0 / C. \qquad (25)$$

We have therefore proved the

Theorem. *If the ellipsoid of inertia at the point O is an ellipsoid of revolution, then the motion of the body is either a rotation about a fixed axis with a constant angular velocity or it is a steady precession.*

Rotation of a body about a point in the general case. We shall now make certain remarks concerning a body which rotates about a point O, under the assumption that the moment of the forces with respect to the point O is zero.

Let us retain the notations used up to the present. The axes ξ, η, ζ, have the directions of the principal axes of inertia at the point O, and hence the equation of the ellipsoid of inertia with respect to O has in the system (ξ, η, ζ) the form (formula (8), p. 164) $A\xi^2 + B\eta^2 + C\zeta^2 = c^2$, where c is an arbitrary constant. Since the kinetic energy E is constant, we can assume $c^2 = 2E$. Hence the ellipsoid of inertia will have the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 = 2E. \qquad (26)$$

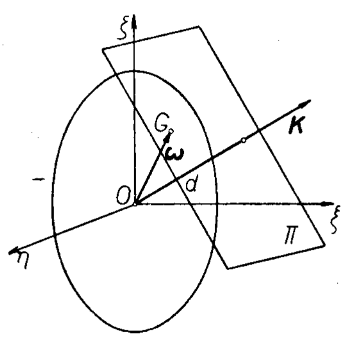

Fig. 300.

Let us denote by G the terminus of the angular velocity vector $\boldsymbol{\omega}$ (Fig. 300). The point G consequently has the coordinates ω_ξ, ω_η, and ω_ζ. By formula (4), p. 400, the coordinates of the point G satisfy equation (26). It follows from this that *the terminus of the vector* $\boldsymbol{\omega}$ *lies on the ellipsoid of inertia* (26).

The equation of the plane Π, tangent to the ellipsoid (26) at the point G, has the form

$$A\omega_\xi \xi + B\omega_\eta \eta + C\omega_\zeta \zeta = 2E. \tag{27}$$

Since by (III), p. 394, the angular momentum $\mathbf{K}$ has the projections $K_\xi = A\omega_\xi$, $K_\eta = B\omega_\eta$, and $K_\zeta = C\omega_\zeta$, on the axes ξ, η, ζ, the angular momentum $\mathbf{K}$ is perpendicular to the plane Π. The distance of the plane Π from the point O is $d = 2E/\sqrt{A^2\omega_\xi^2 + B^2\omega_\eta^2 + C^2\omega_\zeta^2}$; consequently by (1), p. 400,

$$d = 2E \;/\; |\mathbf{K}| = \text{const.} \tag{28}$$

The distance of the plane Π from the point O is therefore constant. Moreover, since the plane Π is constantly perpendicular to the fixed vector $\mathbf{K}$, *Π is a fixed plane in space.*

The ellipsoid of inertia is constantly tangent to the plane Π. The instantaneous motion of the body is an instantaneous rotation with an angular velocity $\boldsymbol{\omega}$, while G is the terminus of the vector $\boldsymbol{\omega}$, and consequently the velocity of the point G is zero. It follows from this that *the ellipsoid of inertia rolls on the plane Π.*

We shall now consider the question, what positions can the vector $\boldsymbol{\omega}$ assume in the body, i. e. what curve does the point G describe on the ellipsoid of inertia.

Let us denote the coordinates of the point G by ξ, η, ζ. Consequently $\xi = \omega_\xi$, $\eta = \omega_\eta$, and $\zeta = \omega_\zeta$. Hence by (1), p. 400, we have

$$A^2\xi^2 + B^2\eta^2 + C^2\zeta^2 = K^2, \tag{29}$$

where $K = |\mathbf{K}|$. The coordinates of the point G also satisfy equation (26) of the ellipsoid of inertia. Multiplying both sides of equation (26) by K^2, and of equation (29) by $2E$, and subtracting, we obtain

$$(AK^2 - 2EA^2)\,\xi^2 + (BK^2 - 2EB^2)\,\eta^2 + (CK^2 - 2EC^2)\,\zeta^2 = 0. \tag{30}$$

Equation (30) is the equation of a cone with its vertex at O. The point G therefore describes a curve which is the intersection of the ellipsoid of inertia (26) and the cone (30). These curves are closed and (in general) of the fourth degree. In particular, the cone (30) is a cone of revolution when, e. g. $A = B$ (or $A = C$ or $B = C$). If A, B, and C, are different, the cone can degenerate into two planes.

The angular velocities traced in the body form the cone (30). Consequently the cone defined by equation (30) is the moving cone of instantaneous angular velocities (p. 339).

If we trace the positions of the point G on the fixed plane Π, then we shall obtain a certain curve. The cone for which this curve is the directrix, and O the vertex, is the fixed cone of instantaneous angular velocities (p. 339).

§ 8. Rotation of a heavy body about a point. We shall now consider the motion of a heavy body in which an arbitrary point O other than the centre of gravity is fixed.

Such a motion is executed, for example, by a top rotating about an axis one of whose ends rests on a sufficiently rough floor, so that the sliding of the end of the axis on the floor is impossible.

For simplicity's sake, let us assume that the body has an axis of symmetry passing through O (Fig. 301). The centre of mass S obviously lies on this axis. Let us take the axis of symmetry as the ζ-axis of the moving coordinate system and give it a sense towards the centre of mass S. Let us put $OS = l$. Let the z-axis of the fixed coordinate system have a sense vertically upwards. The weight of the body $\mathbf{Q}$ therefore has the direction of the z-axis. Denoting by $\mathbf{k}$ the unit vector lying on the z-axis, and by m the mass of the body, we have $\mathbf{Q} = -mg\mathbf{k}$. Forming the projections on the axes ξ, η, ζ, we obtain by (24), p. 356:

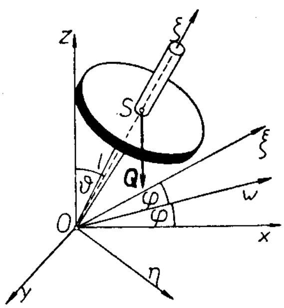

Fig. 301.

$$Q_\xi = -mg\sin\vartheta\sin\varphi,\quad Q_\eta = mg\sin\vartheta\cos\varphi,\quad Q_\zeta = -mg\cos\vartheta.$$

Since $\mathbf{Q}$ has its origin at the centre of mass S whose coordinates in the system (ξ, η, ζ) are $0, 0, l$, denoting by $\boldsymbol{M}$ the moment of the weight $\mathbf{Q}$ with respect to O, we get:

$$M_\xi = mgl\sin\vartheta\cos\varphi,\quad M_\eta = mgl\sin\vartheta\sin\varphi,\quad M_\zeta = 0.$$

As $A = B$, Euler's equations (I), p. 397, assume the form:

$$\begin{aligned} A\dot\omega_\xi + (A - C)\,\omega_\eta\omega_\zeta &= mgl\sin\vartheta\cos\varphi,\\ A\dot\omega_\eta - (A - C)\,\omega_\zeta\omega_\xi &= mgl\sin\vartheta\sin\varphi,\\ C\dot\omega_\zeta &= 0. \end{aligned}\tag{1}$$

If we express ω_ξ, ω_η, ω_ζ, in equations (1) in terms of Euler's angles according to formulae (I), p. 356, we obtain a system of differential equations of the second order, where the unknowns will be ϑ, φ, and ψ, as functions of the time.

System (1) can be reduced to a system of differential equations of the first order in a simple way. We get one equation from the third equation of system (1). Integrating this equation, we obtain $C\omega_\zeta = \text{const}$; hence

$$\omega_\zeta = r = \text{const}. \tag{2}$$

Two other equations of the first order are obtained from the principle of conservation of energy (for the weight possesses a potential) and from the principle of angular momentum, according to which the angular momentum with respect to the fixed axis z is in this case constant, because the moment of the weight with respect to this axis is zero (for the weight has the direction of the z-axis).

The centre of mass has the coordinate $z = l \cos\vartheta$; therefore the potential of the weight is $V = -mgz = -mgl \cos\vartheta$.

The kinetic energy of the body ((4), p. 400) is

$$E = \tfrac{1}{2}[A(\omega_\xi^2 + \omega_\eta^2) + C\omega_\zeta^2],$$

and from the principle of conservation of energy we have $E - V = \text{const}$; consequently

$$A(\omega_\xi^2 + \omega_\eta^2) + C\omega_\zeta^2 + 2mgl \cos\vartheta = h = \text{const}. \tag{3}$$

Denoting by $\mathbf{K}$ the angular momentum with respect to O, we have ((III), p. 394) $K_\xi = A\omega_\xi$, $K_\eta = A\omega_\eta$, and $K_\zeta = C\omega_\zeta$. The z-axis makes with the axes ξ, η, and ζ, angles whose cosines are k_ξ, k_η, and k_ζ (because $\mathbf{k}$ is the unit vector having the sense and the direction of the z-axis). Therefore the projection of the angular momentum $\mathbf{K}$ on the z-axis is $K_z = K_\xi k_\xi + K_\eta k_\eta + K_\zeta k_\zeta$. Substituting into this formula the values k_ξ, k_η, k_ζ, from formulae (24), p. 356, and remembering that $K_z = \text{const}$, because the moment of the weight with respect to the vertical axis z is zero, we obtain

$$K_z = A(\omega_\xi \sin\vartheta \sin\varphi - \omega_\eta \sin\vartheta \cos\varphi) + C\omega_\zeta \cos\vartheta = \text{const}. \tag{4}$$

Let us now express the projections ω_ξ, ω_η, ω_ζ, in formulae (2)—(4) in terms of Euler's angles according to formulae (I), p. 356. We obtain:

$$\begin{gathered} \dot\psi \cos\vartheta + \dot\varphi = r, \quad \dot\vartheta^2 + \dot\psi^2 \sin^2\vartheta + a \cos\vartheta = b, \\ \dot\psi \sin^2\vartheta + \alpha \cos\vartheta = \beta, \end{gathered} \tag{5}$$

where

$$\begin{gathered} r = \omega_\zeta, \quad a = 2mgl \,/\, A, \quad b = (h - Cr^2) \,/\, A, \quad \alpha = Cr \,/\, A, \\ \beta = K_z \,/\, A. \end{gathered} \tag{6}$$

The third of the equations (5) gives $\dot\psi = (\beta - \alpha \cos\vartheta) \,/\, \sin^2\vartheta$. Substituting this value of $\dot\psi$ into the second of the equations (5) and multiplying by $\sin^2\vartheta$, we get

$$\dot\vartheta^2 \sin^2\vartheta + (\beta - \alpha \cos\vartheta)^2 = (b - a \cos\vartheta) \sin^2\vartheta. \tag{7}$$

Let us substitute

$$u = \cos\vartheta \quad \text{or} \quad u^{\cdot} = -\vartheta^{\cdot} \sin\vartheta \tag{8}$$

into (7), and then into the first and third of the equations (5).

We obtain:

$$u^{\cdot 2} = (b - au)(1 - u^2) - (\beta - \alpha u)^2, \tag{9}$$

$$\psi^{\cdot} = (\beta - \alpha u) \,/\, (1 - u^2), \tag{10}$$

$$\varphi^{\cdot} = r - (\beta - \alpha u)\, u \,/\, (1 - u^2). \tag{11}$$

From (9) we can determine u, i. e. $\cos\vartheta$, and then from (10) and (11) the angles ψ and φ.

Let us denote the right side of equation (9) by $f(u)$. Assuming that $\beta - \alpha \neq 0$ and $\beta + \alpha \neq 0$ we have:

$$f(+1) < 0, \quad f(-1) < 0, \tag{12}$$

and in addition, since $a > 0$, by (6)

$$\lim_{u \to +\infty} f(u) = +\infty. \tag{13}$$

If ϑ_0 was the value of the angle ϑ for $t = 0$, and $u_0 = \cos\vartheta_0$, then by (9)

$$f(u_0) = u_0^{\cdot 2} \geqq 0. \tag{14}$$

From relations (12)—(14) it follows that the equation $f(u) = 0$ has three real roots, two of which, namely, u_1 and u_2, lie between -1 and $+1$, and the third $u_3 > 1$. In a particular case we can have $u_1 = u_2$ (a double root).

Let us assume that $u_1 < u_2$. Since $u = \cos\vartheta$ must lie between -1 and $+1$, and moreover $f(u) \geqq 0$ (by (9)), then $u_1 \leqq u \leqq u_2$ or

$$u_1 \leqq \cos\vartheta \leqq u_2. \tag{15}$$

Therefore: *during motion the angle ϑ varies between the limits ϑ_1 and ϑ_2, where $\cos\vartheta_1 = u_1$ and $\cos\vartheta_2 = u_2$. Since $l\cos\vartheta$ denotes the height of the centre of mass above the horizontal xy-plane, the centre of mass oscillates between two horizontal planes $z = l\cos\vartheta_1$ and $z = l\cos\vartheta_2$.*

The numerator of the right member of equation (10) is zero only for $u = \beta / \alpha$. Therefore, if $|\beta / \alpha| > 1$, then the sign of $\psi^{\cdot}$ is constant, because $|u| \leqq 1$. It is obvious that if

$$u_1 < \beta / \alpha < u_2, \tag{16}$$

then $\psi^{\cdot}$ changes its sign. It is easy to show that inequality (16) is equivalent to the inequalities:

$$|\beta / \alpha| < 1, \quad \beta / \alpha < b / a. \tag{17}$$

For if (16) holds, then $|\beta / \alpha| < 1$ and moreover

$$f(\beta / \alpha) = (b - a\beta / \alpha)(1 - \beta^2 / \alpha^2) > 0; \tag{18}$$

hence $b - a\beta / \alpha > 0$, whence $\beta / \alpha < b / a$ (because $a > 0$ by (6)). Conversely, if the inequalities (17) hold, then according to (18) $f(\beta / \alpha) > 0$; hence either inequality (16) or inequality $\beta / \alpha > u_3$ holds. However, the latter is impossible, since $f(b / a) = -(\beta - \alpha b / a)^2 < 0$; hence $b / a < u_3$, and consequently $\beta / \alpha < u_3$ by (17).

§ 9. Motion of a sphere on a plane. Let a heavy sphere of constant density move along a horizontal plane Π (example: the motion of a sphere along a billiard table). Let us consider friction and assume that the reaction of the plane reduces to one force acting at the point of tangency S (Fig. 302). Let us denote by R and T the absolute values of the reactions: normal $\mathbf{R}$ and tangent (friction) $\mathbf{T}$, and by μ the coefficient of friction. Consequently

$$T = \mu R. \tag{1}$$

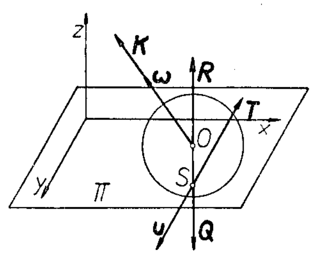

Fig. 302.

If the point of tangency S of the sphere with the plane Π has a velocity different from zero, then the friction $\mathbf{T}$ has the direction of this velocity, but an opposite sense (p. 367). Let us take the plane Π as the xy-plane of the coordinate system (x, y, z) and give the z-axis a sense vertically upwards. Denoting by $\mathbf{p}_0$ the acceleration of the centre of mass $O(x_0, y_0, z_0)$ of the sphere, by m the mass of the sphere, and by $\mathbf{Q}$ its weight, we have

$$m\mathbf{p}_0 = \mathbf{Q} + \mathbf{R} + \mathbf{T}. \tag{2}$$

Forming projections on the coordinate axes, we get:

$$m\ddot{x}_0 = T_x, \; m\ddot{y}_0 = T_y, \; m\ddot{z}_0 = -mg + R. \tag{3}$$

Since $z_0 = \text{const} = r$ (where r denotes the radius of the sphere), $\ddot{z}_0 = 0$, and consequently $R = mg$, and by (1), under the assumption that S has a velocity different from zero,

$$T = \mu mg. \tag{4}$$

Let $\mathbf{K}$ be the angular momentum with respect to the centre O of the sphere, $\boldsymbol{\omega}$ the instantaneous angular velocity, and A the moment of inertia with respect to a diameter. Since by hypothesis the sphere is homogeneous, its centre O is a spherical point. Consequently (by (3), p. 395) $\mathbf{K} = A\boldsymbol{\omega}$, whence

$$\mathbf{K}^{\cdot} = A\boldsymbol{\omega}^{\cdot}. \tag{5}$$

The moment $\mathbf{M}$ of the forces with respect to the point O reduces to the moment of the force $\mathbf{T}$. Consequently:

$$M_x = -rT_y, \quad M_y = rT_x, \quad M_z = 0. \tag{6}$$

Since $\mathbf{K}^{\cdot} = \mathbf{M}$, we get from (5):

$$A\omega_x^{\cdot} = -rT_y, \quad A\omega_y^{\cdot} = rT_x, \quad A\omega_z^{\cdot} = 0. \tag{7}$$

From the last equation we obtain

$$\omega_z = \text{const.} \tag{8}$$

Let $\mathbf{u}$ denote the velocity of the point of tangency S. The instantaneous motion of the sphere is the composition of an advancing motion with a velocity $\mathbf{v}_0$ of the center of mass O and of a rotation about an axis passing through O with an angular velocity $\boldsymbol{\omega}$. Therefore $\mathbf{u} = \mathbf{v}_0 + \overline{OS} \times \boldsymbol{\omega}$, whence:

$$u_x = x_0^{\cdot} + r\omega_y, \quad u_y = y_0^{\cdot} - r\omega_x, \quad u_z = 0. \tag{9}$$

Calculating the derivatives with respect to time, we obtain:

$$u_x^{\cdot} = x_0^{\cdot\cdot} + r\omega_y^{\cdot}, \quad u_y^{\cdot} = y_0^{\cdot\cdot} - r\omega_x^{\cdot}, \quad u_z^{\cdot} = 0, \tag{10}$$

whence by (3) and (7):

$$u_x^{\cdot} = (1 / m + r^2 / A)\, T_x, \quad u_y^{\cdot} = (1 / m + r^2 / A)\, T_y, \quad u_z^{\cdot} = 0. \tag{11}$$

Multiplying both sides of the first of the equations (11) by u_x, and both sides of the second by u_y, and adding, we obtain:

$$u_x u_x^{\cdot} + u_y u_y^{\cdot} = (1 / m + r^2 / A)(T_x u_x + T_y u_y). \tag{12}$$

If $\mathbf{u} \neq 0$, then $\mathbf{T}$ has the direction of $\mathbf{u}$, but an apposite sense. Consequently $\mathbf{Tu} \leqq 0$ constantly, i.e. $T_x u_x + T_y u_y \leqq 0$, from which by (12) $u_x u_x^{\cdot} + u_y u_y^{\cdot} \leqq 0$. Since $u_x u_x^{\cdot} + u_y u_y^{\cdot} = \frac{1}{2}\mathrm{d}(u_x^2 + u_y^2) / \mathrm{d}t$, it follows that $|\mathbf{u}|^2 = u_x^2 + u_y^2$ is a non-increasing function. Therefore, if $\mathbf{u} = 0$ at a certain moment, then from this moment on $\mathbf{u} = 0$ constantly.

Let us assume that, during a certain interval of time, $\mathbf{u}$ was different from zero and the friction $\mathbf{T}$ had the direction of $\mathbf{u}$ (but an opposite sense); we can therefore assume that $\mathbf{T} = \lambda\mathbf{u}$, where $\lambda < 0$ (while λ depends on the time). Hence $\lambda u_x = T_x$ and $\lambda u_y = T_y$, whence by (11)

$$u_x y_y^{\cdot} - u_x^{\cdot} u_y = 0. \tag{13}$$

From equation (13) it follows that $\boldsymbol{u}$ has a constant direction: for putting $u = |\boldsymbol{u}|$ and denoting by φ the angle between $\boldsymbol{u}$ and the x-axis, we get $u_x = u \cos\varphi$, $u_y = u \sin\varphi$, $u_x^{\cdot} = u^{\cdot} \cos\varphi - u\varphi^{\cdot} \sin\varphi$, and $u_y^{\cdot} = u^{\cdot} \sin\varphi + u\varphi^{\cdot} \cos\varphi$; hence $u_x u_y^{\cdot} - u_x^{\cdot} u_y = u^2\varphi^{\cdot}$, whence by (13) $u^2\varphi^{\cdot} = 0$, and since $u^2 \neq 0$, $\varphi^{\cdot} = 0$, i. e. $\varphi = \text{const}$.

Now, since $\boldsymbol{T}$ has the direction of the velocity $\boldsymbol{u}$, the direction of the friction $\boldsymbol{T}$ is also constant. Under the assumption that the coefficient of friction μ is constant (p. 367) we obtain by (4) $T = \text{const}$.

Therefore: *during the entire time in which* $\boldsymbol{u} \neq 0$, $\boldsymbol{T} = \text{const}$.

Since the motion of a material point under the influence of a constant force takes place along a parabola (p. 82), *the centre of mass of the sphere describes a parabola* (whose axis is parallel to the direction of $\boldsymbol{T}$) *during the entire time in which* $\boldsymbol{u} \neq 0$ (i. e. *in which the point of tangency of the sphere with the plane Π has a velocity different from zero*).

Let us assume that at $t = 0$:

$$x_0 = 0, \quad y_0 = 0, \quad z_0 = r, \quad x_0^{\cdot} = a, \quad y_0^{\cdot} = b, \quad z_0^{\cdot} = 0, \qquad (14)$$
$$\omega_x = \omega_x^0, \quad \omega_y = \omega_y^0, \quad \omega_z = \omega_z^0.$$

Hence by (9) the initial velocity $\boldsymbol{u}_0$ of the point of contact S has the projections:

$$u_x^0 = a + r\omega_y^0, \quad u_y^0 = b - r\omega_x^0, \quad u_z^0 = 0. \qquad (15)$$

Let us assume that $\boldsymbol{u}_0 \neq 0$ and give to the y-axis a direction and sense of the velocity $\boldsymbol{u}_0$. Therefore by (15) there will be the following relations among the given initial values:

$$u_x^0 = a + r\omega_y^0 = 0, \quad u_y^0 = b - r\omega_x^0 > 0. \qquad (16)$$

Now, because $\boldsymbol{u}$ and $\boldsymbol{T}$ have the same directions, but opposite senses,

$$T_x = 0, \quad T_y = -\mu mg. \qquad (17)$$

After integration and consideration of the initial conditions, we obtain from equations (3) and (7):

$$x_0 = at, \quad y_0 = -\tfrac{1}{2}\mu g t^2 + bt, \quad z_0 = r, \qquad (18)$$
$$\omega_x = \mu mgrt \,/\, A + \omega_x^0, \quad \omega_y = \omega_y^0, \quad \omega_z = \omega_z^2. \qquad (19)$$

Substituting the values from (18) and (19) in equations (9), we obtain in view of (16):

$$u_x = 0, \quad u_y = (b - r\omega_x^0) - (1 + mr^2 \,/\, A)\, gt\mu, \quad u_z = 0. \qquad (20)$$

Since $b - r\omega_x^0 > 0$ it follows by (16), that after the time

$$t_1 = \frac{b - r\omega_x^0}{(1 + mr^2 \,/\, A)\,\mu g} \qquad (21)$$

$u_x = 0$, $u_y = 0$, and $u_z = 0$, i. e. $\boldsymbol{u} = 0$; and after the time t_1, $\boldsymbol{u} = 0$ constantly. Hence by (11) $T_x = 0$ and $T_y = 0$, i. e. $\boldsymbol{T} = 0$ constantly. From equations (3) and (7) we obtain then:

$$x_0^{\cdot\cdot} = 0, \quad y_0^{\cdot\cdot} = 0, \quad z_0^{\cdot\cdot} = 0, \quad \omega_x^{\cdot} = 0, \quad \omega_y^{\cdot} = 0, \quad \omega_z^{\cdot} = 0.$$

Therefore: *from the time* t_1, $\boldsymbol{v}_0 =$ const. *and* $\omega =$ const. *constantly, i. e. the centre of the sphere will move with a uniform motion along a straight line from the time* t_1 *on, while the instantaneous angular velocity of the sphere will be constant.*

§ 10. Foucault's gyroscope. This is the name we give to a heavy body having an axis of symmetry and suspended at the centre of mass (*the so--called Cardan's suspension*).

Since the force of gravity acts at the center of mass, in the case when no other forces act on the body, the motion of a gyroscope reduces to a rotation of the body about the centre of mass under the action of no forces.

If the body is set spinning about the centre of mass and initially the axis of symmetry is the instantaneous axis of rotation, then the axis of symmetry will maintain a constant direction in space. This follows from the theorem given on p. 401 and from the observation that the axis of symmetry is a central axis of inertia of the body.

It is true that the axis of symmetry will move relative to the earth, however, this will only be an apparent motion (induced by the rotation of the earth): for if the axis of symmetry is directed towards some fixed star, then the axis will point to it constantly.

We shall consider here the cases in which the axis of symmetry is not free, but is confined either to a meridional plane or to a horizontal plane.

Motion of the axis of symmetry in a meridional plane. Let it be possible for a body (suspended at the centre of mass) to move only in a meridional plane passing through a given point on the earth. We can assume that the forces (reactions) holding the axis in the meridional plane are perpendicular to this plane and have their points of application on the axis of symmetry.

Let us denote by $\boldsymbol{\omega}_1$ the angular velocity vector of the earth and set:

$$\omega_1 = |\boldsymbol{\omega}_1|. \tag{1}$$

Let us take the centre of mass O of the body as the origin of the coordinate system (x, y, z), giving the z-axis the direction and sense of the angular velocity $\boldsymbol{\omega}_1$ of the earth, and the x-axis a horizontal direction with a sense towards the east (Fig. 303).

The yz-plane will consequently be a meridional plane, and at a given place the z-axis will make an angle of $90° - \varphi$ with the vertical, where φ denotes the latitude of this place.

In addition, let us select a second coordinate system (ξ, η, ζ) whose origin is at O, taking the axis of symmetry of the body as the ζ-axis, and the x-axis as the ξ-axis. The plane $\eta\zeta$ will therefore be identical with the meridional plane yz. The position of the system (ξ, η, ζ) is defined by the angle ϑ which the axes ζ and z make with each other (where the angle ϑ is defined as the angle through which it is necessary to rotate the z-axis from right to left with respect to the x-axis, in order that the positive directions of the axes z and ζ coincide with each other).

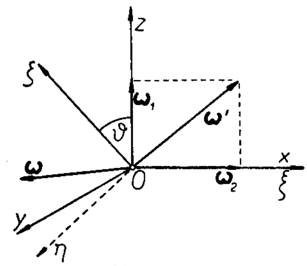

Fig. 303.

Let $\boldsymbol{\omega}'$ be the instantaneous angular velocity of the system (ξ, η, ζ) with respect to an inertial frame, which we take to be a frame attached to the sun and the fixed stars. It is easy to see that $\boldsymbol{\omega}'$ is the resultant of the instantaneous angular velocity $\boldsymbol{\omega}_2$ of the system (ξ, η, ζ) relative to (x, y, z) and of the angular velocity $\boldsymbol{\omega}_1$ of the system (x, y, z) relative to the inertial frame. Consequently

$$\boldsymbol{\omega}' = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2. \tag{2}$$

Since the vector $\boldsymbol{\omega}_1$ has by hypothesis the direction and sense of the z-axis, its projections on the axes of the system (ξ, η, ζ) are in virtue of (1):

$$\omega_{1_\xi} = 0, \quad \omega_{1_\eta} = -\omega_1 \sin\vartheta, \quad \omega_{1_\zeta} = \omega_1 \cos\vartheta. \tag{3}$$

The system (ξ, η, ζ) rotates about the ξ-axis relative to the system (x, y, z). Since the angle of rotation is ϑ, the instantaneous angular velocity has the direction of the ξ-axis and its component with respect to the ξ-axis is $\vartheta^{\cdot}$. Consequently:

$$\omega_{2_\xi} = \vartheta^{\cdot}, \quad \omega_{2_\eta} = 0, \quad \omega_{2_\zeta} = 0. \tag{4}$$

In virtue of (2) — (4):

$$\omega'_\xi = \vartheta^{\cdot}, \quad \omega'_\eta = -\omega_1 \sin\vartheta, \quad \omega'_\zeta = \omega_1 \cos\vartheta. \tag{5}$$

Let $\boldsymbol{\omega}$ denote the instantaneous angular velocity of a body relative to the inertial frame. The vector $\boldsymbol{\omega}$ can be considered as the composition of the instantaneous angular velocity $\boldsymbol{\omega}_3$ of the body relative to the system (ξ, η, ζ) and the velocity $\boldsymbol{\omega}'$ of the system (ξ, η, ζ) relative to the inertial frame. Therefore $\boldsymbol{\omega} = \boldsymbol{\omega}_3 + \boldsymbol{\omega}'$. Since the motion of the body relative to (ξ, η, ζ) is a rotation about the ζ-axis, it follows that $\omega_{3_\xi} = 0$ and $\omega_{3_\eta} = 0$, whence:

$$\omega_\xi = \omega'_\xi, \quad \omega_\eta = \omega'_\eta, \quad \omega_\zeta = \omega_{3_\zeta} + \omega'_\zeta \tag{6}$$

(in addition, since ω_1 is very small, by (5) ω'_ζ is also small; hence for all practical purposes $\omega_\zeta = \omega_{3_\zeta}$). Putting $\omega_\zeta = \omega$, we obtain by (5) and (6):

$$\omega_\xi = \vartheta^\cdot, \quad \omega_\eta = -\omega_1 \sin\vartheta, \quad \omega_\zeta = \omega. \tag{7}$$

The axes ξ, η, ζ, are the central axes of inertia of the body, because ζ is the axis of symmetry and O the centre of mass. Denoting the angular momentum with respect to O by $\mathbf{K}$, the moments of inertia with respect to ξ and η by A, and the moment of inertia with respect to ζ by C, we obtain by (III), p. 394, and (7):

$$K_\xi = A\vartheta^\cdot, \quad K_\eta = -A\omega_1 \sin\vartheta, \quad K_\zeta = C\omega, \tag{8}$$

whence after differentiation:

$$K^\cdot_\xi = A\vartheta^{\cdot\cdot}, \quad K^\cdot_\eta = \; A\omega_1\vartheta^\cdot \cos\vartheta, \quad K^\cdot_\zeta = C\omega^\cdot. \tag{9}$$

The moment of the weight with respect to O is zero. The moment of the forces holding the axis of symmetry of the body in the plane of the meridian is zero with respect to the axes ξ and ζ, because these forces have their points of application on the ζ-axis and are parallel to the ξ-axis. Therefore, denoting by $\boldsymbol{M}$ the moment of the forces with respect to the centre of mass O, we obtain:

$$M_\xi = 0, \quad M_\zeta = 0. \tag{10}$$

To determine the motion of the body we apply equations (II), p. 398. From these equations, after substituting in them the values from (5), (8), (9), (10), and after reducing, we obtain:

$$\begin{aligned} &A\vartheta^{\cdot\cdot} - A\omega_1^2 \sin\vartheta \cos\vartheta + C\omega\omega_1 \sin\vartheta = 0, \\ &-2A\omega_1\vartheta^\cdot \cos\vartheta + C\omega\vartheta^\cdot = M_\eta, \; C\omega^\cdot = 0. \end{aligned} \tag{I}$$

In virtue of the last equation $\omega = \text{const}$. Dropping the term containing ω_1^2 from the first of the equations (I), because it is very small, we obtain

$$\vartheta^{\cdot\cdot} = -\frac{C\omega\omega_1}{A} \sin\vartheta. \tag{11}$$

Since $\omega = \text{const}$, we can give the ζ-axis a sense such that $\omega > 0$ constantly, i. e. such that the rotation of the body relative to the axis of symmetry ζ is from right to left. Under this assumption $C\omega\omega_1 / A > 0$. Equation (11) therefore has the form of the differential equation for the simple pendulum (p. 130, formula (I)). The positions of equlibrium occur for $\vartheta = 0$ and $\vartheta = \pi$.

The axis of symmetry of the body will therefore oscillate about the z-axis, i. e. about a line parallel to the axis of the earth. The axis of the body can be at rest only for $\vartheta = 0$ or for $\vartheta = \pi$, i. e. only when it is parallel to the axis of the earth. Therefore, determining the position of equilibrium of the axis of the body, we obtain the direction of the axis of the earth. Since the axis of the earth makes an angle of $90° - \varphi$ with the vertical at a given place, *we can in this manner obtain the latitude φ of the given place.*

It can be shown that $\vartheta = 0$ is the position of stable equilibrium, and $\vartheta = \pi$ that of unstable equilibrium. Hence the ζ-axis tends to assume a position such that its direction and sense agree with the direction of the axis of the earth and the sense of the vector $\boldsymbol{\omega}_1$. From formula (3), p. 130, it follows that the period of oscillation of the axis of the body (when the initial angle ϑ_0 is small and $\dot{\vartheta}_0 = 0$) is

$$T = 2\pi\sqrt{A / C\omega\omega_1}. \qquad (12)$$

The period of oscillation is large because ω_1 is small (ca 0.00007 $\sec^{-1}$). However, we can decrease it by increasing ω, i. e. by spinning the body faster about its own axis of symmetry.

Motion of the axis in a horizontal plane. Let us now assume that the axis of symmetry of a body can move only in a horizontal plane. We can therefore assume that the reactions holding the axis horizontally have their points of application on this axis and have a vertical direction.

Let us choose two systems of coordinates (x, y, z) and (ξ, η, ζ) whose common origin is at the center of mass O of the body (Fig 304). Let us give the y-axis a sense vertically upwards, the x-axis a sense towards the east, and the z-axis towards the north. Let us take the axis of symmetry of the body as the ζ-axis and the y-axis as the η-axis. Therefore the $\xi\zeta$ plane will be constantly horizontal. The position of the system (ξ, η, ζ) is defined by the angle ϑ between the axes ζ and z (where ϑ is defined as the angle through which it is necessary to rotate the z-axis about the y-axis from right to left, in order that the positive directions of the axes ζ and z coincide with each other).

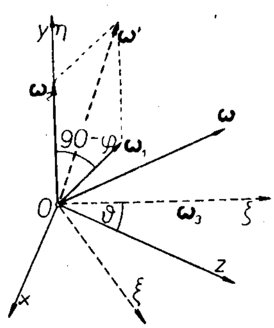

Fig. 304.

The instatanenous angular velocity of the system (x, y, z) with respect to the inertial frame is equal to $\boldsymbol{\omega}_1$ (i. e. to the angular velocity of the earth). The vector $\boldsymbol{\omega}_1$ lies in the yz plane and makes an angle of $90° - \varphi$ with the y-axis (where φ denotes the latitude of the given place). Let $\omega_1 = |\boldsymbol{\omega}_1|$. The projections of $\boldsymbol{\omega}_1$ on the axes ξ, η, ζ, are therefore:

$$\omega_{1_\xi} = \omega_1 \cos\varphi \sin\vartheta, \ \omega_{1_\eta} = \omega_1 \sin\varphi, \ \omega_{1_\zeta} = \omega_1 \cos\varphi \cos\vartheta. \tag{13}$$

The instantaneous angular velocity $\boldsymbol{\omega}_2$ of the system (ξ, η, ζ) with respect to the system (x, y, z) is equal to $\vartheta^\cdot$ and has the direction of the η-axis, because (ξ, η, ζ) rotates about η relative to (x, y, z). Consequently, the instantaneous angular velocity $\boldsymbol{\omega}'$ of the system (ξ, η, ζ) with respect to the inertial frame is the composition of the angular velocity $\boldsymbol{\omega}_2$ and the angular velocity $\boldsymbol{\omega}_1$, whence by (13):

$$\omega'_\xi = \omega_1 \cos\varphi \sin\vartheta, \ \omega'_\eta = \vartheta^\cdot + \omega_1 \sin\varphi, \ \omega'_\zeta = \omega_1 \cos\varphi \cos\vartheta. \tag{14}$$

Let $\boldsymbol{\omega}$ denote the instantaneous angular velocity of a body relative to the inertial frame. Since the instantaneous motion of the body relative to the system (ξ, η, ζ) is an instantaneous rotation about the ζ-axis, the projections of the instantaneous angular velocity $\boldsymbol{\omega}_3$ of the body relative to the system (ξ, η, ζ) on the axes of this system are: $\omega_{3_\xi} = 0$, $\omega_{3_\eta} = 0$. Since $\boldsymbol{\omega} = \boldsymbol{\omega}' + \boldsymbol{\omega}_3$, we obtain by (14) (putting $\omega_\zeta = \omega$):

$$\omega_\xi = \omega_1 \cos\varphi \sin\vartheta, \quad \omega_\eta = \vartheta^\cdot + \omega_1 \sin\varphi, \quad \omega_\zeta = \omega. \tag{15}$$

Denoting the angular momentum with respect to O by $\mathbf{K}$, the moments of inertia with respect to the axes ξ and η by A, and with respect to ζ by C, we obtain by (III), p. 394, and (15):

$$K_\xi = A\omega_1 \cos\varphi \sin\vartheta, \quad K_\eta = A(\vartheta^\cdot + \omega_1 \sin\varphi), \quad K_\zeta = C\omega, \tag{16}$$

whence by differentiation:

$$K^\cdot_\xi = A\omega_1\vartheta^\cdot \cos\varphi \cos\vartheta, \quad K^\cdot_\eta = A\vartheta^{\cdot\cdot}, \quad K^\cdot_\zeta = C\omega^\cdot. \tag{17}$$

As the reactions holding the axis of the body in the horizontal plane have their points of application on the ζ-axis of the body and are perpendicular to $\xi\zeta$, denoting by $\mathbf{M}$ the moment of the acting forces, we obtain:

$$M_\eta = 0, \quad M_\zeta = 0. \tag{18}$$

From formulae (II), p. 398, we obtain after substituting the values from (17), (16), (14), and (18):

$$\begin{gathered} A\omega_1\vartheta^\cdot \cos\varphi \cos\vartheta + A(\vartheta^\cdot + \omega_1 \sin\varphi)\, \omega_1 \cos\varphi \cos\vartheta - \\ - C\omega(\vartheta^\cdot + \omega_1 \sin\varphi) = M_\xi, \\ A\vartheta^{\cdot\cdot} + C\omega\omega_1 \cos\varphi \sin\vartheta - A\omega_1^2 \cos^2\varphi \cos\vartheta \sin\vartheta = 0, \ C\omega^\cdot = 0. \end{gathered} \tag{II}$$

In virtue of the last of the equations (II) $\omega = \text{const}$. Dropping the term ω_1^2 in the second of these equations, because it is very small, we get

$$\vartheta^{\cdot\cdot} = -\frac{C\omega\omega_1 \cos\varphi}{A} \sin\vartheta. \tag{19}$$

Let us give the ζ-axis a sense so that $\omega > 0$. Then

$$C\omega\omega_1 \cos\varphi \,/\, A > 0$$

and equation (19) assumes the form of the equation of the simple pendulum ((I), p. 130). The positions of equilibrium occur for $\vartheta = 0$ and $\vartheta = \pi$.

The axis of symmetry of the body will therefore oscillate about the z-axis, i. e. about a horizontal axis running from south to north. The axis of the body can be at rest only for $\vartheta = 0$ or $\vartheta = \pi$, i. e. only when it lies in a meridional plane. Consequently, determining the position of equilibrium of the axis of a body, *we obtain the direction of the meridian: the body can therefore be used as a compass.*

Let us still note that, as before, $\vartheta = 0$ corresponds to stable equilibrium, and $\vartheta = \pi$ to unstable equilibrium.

The period of oscillation is obtained from formula (3), p. 130:

$$T = 2\pi\sqrt{A \,/\, C\omega\omega_1 \cos\varphi}. \tag{20}$$

It will be smallest on the equator (i. e. for $\varphi = 0$). On the pole, however (i. e. for $\varphi = 90°$), every position will be a position of equilibrium, as follows from formula (19). For when $\varphi = 90°$, $\vartheta^{\cdot\cdot} = 0$; as one comes closer to the pole $T \to \infty$.

The results obtained found confirmation in experiments which proves the earth's rotation about an axis. Such experiments were first performed by L. FOUCAULT.