CHAPTER IX

PRINCIPLE OF VIRTUAL WORK

§ I. Holonomo-scleronomic systems. In the case of the equilibrium of a constrained system of material points or of rigid bodies, in which friction does not appear, the constraints are independent of the time and usually depend on the fact that only certain positions of the system are possible and others impossible.

Examples are: a material point constrained to remain on a fixed curve or surface, a system of two material points connected by a rigid massless wire, a rigid body having a fixed point or axis, a system of rigid bodies tangent to one another or joint-connected, etc.

The constraints of a system can be induced in various ways: e. g. by means of rigid bodies, supports, etc. It turns out, however, that in the case when there is no friction, the conditions of equilibrium of the acting forces do not depend on the origin of the constraints, but only on what positions are possible. Moreover, if it is a matter of investigating the equilibrium of a system, then it is sufficient to know only those positions compatible with the constraints which are near the position investigated.

We shall first consider the manner in which it is possible to represent the position of a system compatible with the constraints. We shall first study this matter by means of examples.

BILATERAL CONSTRAINTS

Example I. Let us assume that a material point is constrained to remain on a certain surface S. The constraints can be defined by giving the equation of this surface, for example, in the form

$$F(x, y, z) = 0. \tag{1}$$

Only those positions of the point will be possible in which the coordinates x, y, z, satisfy equation (1). To investigate the equilibrium of the point at some position $A(x, y, z)$ it is sufficient if (1) is the equation of an element of the surface on which this point lies.

Example 2. Let us assume that a material point is constrained to remain on the curve C whose equations are:

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0; \tag{2}$$

then the coordinates of the point must satisfy the equations (2). To examine the conditions for the equilibrium of the point it is sufficient if the equations (2) represent only an arc of the curve C within which the material point lies.

Example 3. If a system consisting of two points A_1 and A_2 is a rigid system (i. e. the distance of the points $A_1A_2 = \text{const.} = d$), then the coordinates x_1, y_1, z_1, and x_2, y_2, z_2, of these points must satisfy the equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - d^2 = 0. \tag{3}$$

If in addition, for example, the sum of the distances of the points from the origin O of the coordinate system is constant and equal to h, then the coordinates of the points must also satisfy the equation

$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} - h = 0. \tag{4}$$

Example 4. If a system composed of n points:

$$A_1(x_1, y_1, z_1),\ A_2(x_2, y_2, z_2),\ \ldots,\ A_n(x_n, y_n, z_n)$$

is a rigid system (i. e. such that the mutual distances of its points are constant), then the coordinates of each pair of points A_i, A_j must satisfy the equation

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - r_{ij}^2 = 0, \tag{5}$$

where $r_{ij} = A_iA_j$. There are as many equations (5) as there are pairs of points, i. e. $\frac{1}{2}n(n-1)$.

In examples 1—4 the constraints were expressed by equations of the form

$$F(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \tag{6}$$

where F is a function defined in a certain region of the variables $x_1, \ldots, z_n$ and is independent of the time t.

The constraints represented by equations (6) are called *holonomic bilateral constraints independent of the time.*

UNILATERAL CONSTRAINTS

Example 5. A point is to remain within the sphere $x^2 + y^2 + z^2 - r^2 = 0$ or on its surface (as e. g. a material point attached to an inextensible string of length r and suspended from the origin of the system). Hence the coordinates of the point must satisfy the inequality

$$x^2 + y^2 + z^2 - r^2 \leqq 0, \tag{7}$$

which states that the distance of the material point from the origin of the coordinate system is not greater than r.

Example 6. Two material points A_1, A_2 are connected by an inextensible string of length h passing through the origin O of the coordinate system. Consequently $OA_1 + OA_2 \leqq h$. The coordinates of the points therefore satisfy the inequality

$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} - h \leqq 0. \tag{8}$$

In examples 5 and 6 it was possible to express the constraints by inequalities of the form

$$\Phi(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) \leqq 0. \tag{9}$$

Constraints defined by inequalities of the form (9) are called *holonomic unilateral constraints independent of the time.*

Remark. If the constraints are given by the inequality

$$\Psi(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) \geqq 0,$$

then putting $\Psi = -\Phi$, we obtain the form (9) after changing the sign.

The position of a system in which the relation (9) assumes the form of an equality (i. e. in which $\Phi = 0$) is called a *boundary position.*

In examples 5 and 6 the boundary position occurs when the string is in tension.

The constraints of a system can consist simultaneously of bilateral and unilateral constraints of the form (6) and (9). For example, if a material point is constrained to remain on the upper half of the sphere $x^2 + y^2 + z^2 - r^2 = 0$, then the coordinates of the point must satisfy the relations:

$$x^2 + y^2 + z^2 - r^2 = 0, \ z \geqq 0 \ \ (\text{or } -z \leqq 0). \tag{10}$$

A system whose constraints can be represented by means of relations of the form (6) or (9) is called a *holonomo-scleronomic* system. We say that the relations (6) and (9) represent the *constraints in a finite form.*

In this chapter we shall deal exclusively with holonomo-scleronomic systems.

Degrees of freedom of a system. When we investigate the equilibrium of a system, we assume in general that for positions close to those investigated, the bilateral constraints consist in the fact that the coordinates of the points of the system must satisfy the equations:

$$\begin{aligned} &F_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &F_m(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \end{aligned} \tag{11}$$

which we write compactly as

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m). \tag{I}$$

In general, we assume that the functions F_j are continuous and have continuous partial derivatives of the first and second order in a certain region of the values $x_1, \ldots, z_n$, defining the given position of the system.

Moreover, we assume that these functions are *independent* in this region, i. e. that no one of them is a function of those remaining.

We shall finally assume that $m < 3n$. For were $m = 3n$, then the system of equations (I) would have in general only one solution and hence only one position of the body would be possible. On the other hand, were $m > 3n$, then the system of equations (I) would not have in general any solution, since the number of unknowns would be smaller than the number of equations.

The number $k = 3n - m$ is called the *number of degrees of freedom.*

Knowing k variables from among the variables $x_1, \ldots, z_n$, we can calculate from equations (I) the remaining variables: their number is $3n - k = m$.

If a system is free, then $m = 0$. i. e. $k = 3n$. In particular, a free material point therefore has three degrees of freedom. We can then choose all three of its coordinates arbitrarily.

On the other hand, if a point is constrained to remain on a surface (as in example 1), then its coordinates have to satisfy only the equation (1). Consequently $m = 1$, i. e. $k = 3 \cdot 1 - 1 = 2$. The point constrained to the surface therefore has two degrees of freedom. Then one of its coordinates depends on the remaining two.

Finally, if the point is constrained to a curve (as in example 2), then its coordinates must satisfy the two equations (2). Consequently $k = 3 \cdot 1 - 2 = 1$ and the point has only one degree of freedom: knowing one of its coordinates, we can determine the remaining two.

A rigid system of two points (example 3) has $3 \cdot 2 - 1 = 5$ degrees of freedom. Nevertheless, if this system is to satisfy both equations (3) and (4), then $k = 3 \cdot 2 - 2 = 4$.

Generally, let a rigid system consist of n material points (as in example 4). The coordinates of these points must satisfy $\frac{1}{2}n(n-1)$ equations (5). However, these equations are not independent of each other for $n > 3$.

Let us note that the position of a rigid system is determined by giving the position of three of its non-collinear points (e. g. A_1, A_1, A_3), (p. 313). Hence, knowing the coordinates $x_1, y_1, z_1, x_2, y_2, z_2$, and x_3, y_3, z_3, of the points A_1, A_2, A_3, we shall be able to calculate the coordinates of the points $A_4, A_5, \ldots, A_n$, from equations (5).

Among the coordinates of the points A_1, A_2, A_3, there are three equations of the form (5), expressing the fact that the distances A_1A_2, A_1A_3, and A_2A_3, are constant magnitudes. From these equations we can in general determine three unknown coordinates, if we know the six remaining ones. We see, therefore, that if we know a certain six of the $3n$ coordinates $x_1, \ldots, z_n$, we can calculate the remaining ones from the equations (5). Consequently the number of degrees of freedom is $k = 6$.

Therefore: *a rigid system of points has six degrees of freedom.*

The number of independent equations is $m = 3n - k$; hence $m = 3n - 6$. From among $\frac{1}{2}n(n-1)$ equations (5) there are therefore only $3n - 6$ independent ones.

§ 2. Virtual displacements. Point on a surface. Let us assume that a material point is to remain constantly on a certain surface S and that it is at the point A of this surface.

Let us displace the point from position A to position B. The displacement $\overline{AB}$ is said to be *possible* if B also lies on the surface S. In the contrary case the displacement $\overline{AB}$ is called an *impossible displacement.*

If a material point at A is given a velocity $\mathbf{v}$, then this velocity is said to be *possible* or *compatible with the constraints* when the point can possess it while moving on the surface. In the contrary case this velocity is said to be *impossible* or *incompatible with the constraints.*

It is easy to see that every vector tangent to a surface at the point A represents a possible velocity. Conversely, possible velocities are tangent to the surface.

An important role is played by displacements proportional to possible velocities, i. e. those that can be represented by vectors equal to possible velocity vectors.

Displacements proportional to velocities possible at the point A are called *virtual displacements* at this point.

A virtual displacement therefore has a direction tangent to the surface, but an arbitrary sense and magnitude. In general, virtual displacements are not possible displacements. However, they are possible displacements when the surface S is a plane, for instance.

Let the surface S have the equation

$$F(x, y, z) = 0. \tag{1}$$

If a material point is at the point A of the surface S, then its coordinates x, y, z, satisfy equation (1). Let us suppose that the material point moves on the surface S in anentirely arbitrary manner. Equation (1) is therefore satisfied constantly. Differentiating (1) with respect to the time t, we obtain

$$\frac{\partial F}{\partial x} x^{\cdot} + \frac{\partial F}{\partial y} y^{\cdot} + \frac{\partial F}{\partial z} z^{\cdot} = 0. \tag{2}$$

Denoting by $\mathbf{v}$ the velocity of the point, we have $v_x = x^{\cdot}$, $v_y = y^{\cdot}$, $v_z = z^{\cdot}$; consequently

$$\frac{\partial F}{\partial x} v_x + \frac{\partial F}{\partial y} v_y + \frac{\partial F}{\partial z} v_z = 0. \tag{3}$$

Hence we see that the possible velocities must satisfy equation (3). The partial derivatives appearing in this equation are proportional to the direction numbers of the normal to the surface at the point A. Equation (3) therefore expresses the fact that the velocity $\mathbf{v}$ is perpendicular to the normal, i. e. that it lies in the tangent plane.

Conversely, if some velocity satisfies equation (3), then it is a possible velocity.

Let us denote by $\overline{\delta s}$ an arbitrary displacement of the point A, and by δx, δy, δz, the projections of this displacement on the coordinate axes (Fig. 305). According to the definition, the virtual displacement is proportional to a possible velocity $\mathbf{v}$. The displacement $\overline{\delta s} = \mathbf{v}$ will consequently be a virtual displacement.

In virtue of (3) the projections of the virtual displacement therefore satisfy the equation

$$\frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial z} \delta z = 0. \tag{4}$$

Conversely, if the projections of some vector $\overline{\delta s}$ satisfy equation (4), then $\overline{\delta s}$ is a virtual displacement.

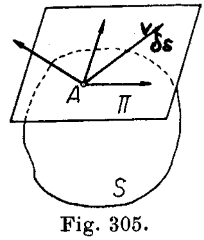

Fig. 305.

We can therefore say that *a virtual displacement is a vector whose projections satisfy equation* (4).

Example 1. A point is constrained to lie on the sphere $x^2 + y^2 + z^2 - r^2 = 0$. Let $A(x, y, z)$ be an arbitrary point on this sphere. The virtual displacements at the point A satisfy the equation

$$2x\,\delta x + 2y\,\delta y + 2z\,\delta z = 0, \text{ whence } x\,\delta x + y\,\delta y + z\,\delta z = 0.$$

Assuming e. g. that $z \neq 0$, we obtain

$$\delta z = -(x\,\delta x + y\,\delta y)\,/\,z. \qquad (5)$$

Choosing arbitrary δx, δy, and taking the value δz from (5), we get a set of numbers δx, δy, δz, representing the projections of the virtual displacement at the point A.

Point on a curve. Suppose that a material point is constrained to remain on a fixed curve L defined by the equations:

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0. \qquad (6)$$

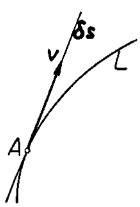

Fig. 306.

If the material point is at the point A, then the coordinates x, y, z, of this point satisfy equations (6). It is easy to see that the material point can have only those velocities whose directions are tangent to the curve L at the point A (Fig. 306). By definition, therefore, the virtual displacements have directions tangent to the curve, but arbitrary senses and lengths.

If the point moves along the curve L, equations (6) are satisfied constantly. Differentiating them, we get:

$$\frac{\partial F_1}{\partial x} x^{\cdot} + \frac{\partial F_1}{\partial y} y^{\cdot} + \frac{\partial F_1}{\partial z} z^{\cdot} = 0, \quad \frac{\partial F_2}{\partial x} x^{\cdot} + \frac{\partial F_2}{\partial y} y^{\cdot} + \frac{\partial F_2}{\partial z} z^{\cdot} = 0. \qquad (7)$$

Denoting by $\overline{\delta s}$ the virtual displacement, and by δx, δy, δz, its projections, we can assume according to the definition $\overline{\delta s} = \mathbf{v}$, whence $\delta x = v_x = x^{\cdot}$, etc. Hence in virtue of (7):

$$\frac{\delta F_1}{\partial x}\,\delta x + \frac{\partial F_1}{\partial y}\,\delta y + \frac{\partial F_1}{\partial z}\,\delta z = 0, \quad \frac{\partial F_2}{\partial x}\,\delta x + \frac{\partial F_2}{\partial y}\,\delta y + \frac{\partial F_2}{\partial z}\,\delta z = 0. \qquad (8)$$

Conversely, if some displacement $\overline{\delta s}$ satisfies equations (8), then it is a virtual displacement.

Example 2. A point is constrained to a curve defined by equations:

$$x^2 + y^2 + z^2 - r^2 = 0, \quad x^2 + 2y^2 - z = 0. \qquad (9)$$

Let the point $A(x, y, z)$ lie on this line. The virtual displacement $\overline{\delta s}$ at the point A therefore satisfies the equations:

$$x\,\delta x + y\,\delta y + z\,\delta z = 0, \quad 2x\,\delta x + 4y\,\delta y - \delta z = 0. \tag{10}$$

If $x \neq 0$ and $y \neq 0$, then we get from (10):

$$\delta x = -(1 + 4z)\,\delta z\,/\,2x, \quad \delta y = (1 + 2z)\,\delta z\,/\,2y.$$

Hence, selecting δz arbitrarily and then determining δx, δy, from the last equations, we obtain a set of numbers δx, δy, δz, defining the virtual displacement at A.

On the other hand, if $x = 0$, for example, then because $z \geqq 0$ (which follows from the second of the equations (9)), we obtain by (10) $\delta y = 0$ and $\delta z = 0$. In this case the virtual displacement will consequently have the projections δx, 0, 0, where δx is an arbitrary number. Therefore: the virtual displacement has the direction of the x-axis.

Holonomo-scleronomic systems. Let us now define virtual displacements in the general case.

Let there be given a holonomo-scleronomic system of n material points $A_1, A_2, \ldots, A_n$.

The system of vectors $\overline{A_1B_1}, \overline{A_2B_2}, \ldots, \overline{A_nB_n}$ (representing the displacements of the individual points), is called briefly a *displacement of the system*. A displacement of the system of points $A_1, A_2, \ldots, A_n$ is said to be *possible*, if the final positions $B_1, B_2, \ldots, B_n$ are compatible with the constraints. In the contrary case the displacement of the system is said to be *impossible*.

Let us give a system of points in the position $A_1, A_2, \ldots, A_n$ the arbitrary velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. A system of these velocities is said to be a *system of possible velocities* if the points can have these velocities and move compatible with the constraints. In the contrary case the system of velocities is said to be *impossible*.

A *virtual displacement* of a system of material points in a certain position is said to be a displacement in which the separate points experience displacements proportional to the system of possible velocities.

Therefore, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, is a system of possible velocities, then the virtual displacement of a system is obtained by giving the points of the system the displacements:

$$\overline{\delta s_1} = \mathbf{v}_1, \ \overline{\delta s_2} = \mathbf{v}_2, \ldots, \overline{\delta s_n} = \mathbf{v}_n. \tag{11}$$

Let us note that if a system is free, then every displacement of the system is a virtual displacement, because every system of velocities is a possible system.

We shall now consider the determination of the virtual displacements. We shall first discuss the case of bilateral constraints and then that of the unilateral constraints.

Bilateral constraints. Let us assume that the constraints are defined by the equations:

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m), \tag{12}$$

which must be satisfied by the coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n$, of the points of the system. Let us give the system an arbitrary motion compatible with the constraints. Differentiating equations (12), we get:

$$\frac{\partial F_j}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial F_j}{\partial z_n}\dot{z}_n = 0 \qquad (j = 1, 2, \ldots, m). \tag{13}$$

Denoting by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ the velocities of the points of the system we therefore obtain $v_{1_x} = \dot{x}_1, \ldots, v_{n_z} = \dot{z}_n$, whence

$$\frac{\partial F_j}{\partial x_1}v_{1_x} + \ldots + \frac{\partial F_j}{\partial z_n}v_{n_z} = 0 \qquad (j = 1, 2, \ldots, m). \tag{14}$$

Consequently every possible system of velocities must satisfy equations (14). Conversely, it is possible to show that if a system of velocities satisfies equations (14), then it is a possible system of velocities.

Let us assume that the displacement of a system, in which the displacements of the successive points $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, is a virtual displacement. The velocities $\mathbf{v}_1, \ldots, \mathbf{v}_n$, defined by equations (11) therefore form a possible system of velocities, in view of which the equations (14) are satisfied. Denoting the projections of the displacements by $\delta x_1, \ldots, \delta z_n$, respectively, we obtain from (14)

$$\frac{\partial F_j}{\partial x_1}\delta x_1 + \ldots + \frac{\partial F_j}{\partial z_n}\delta z_n = 0 \qquad (j = 1, 2, \ldots, m) \tag{15}$$

or, written differently,

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\delta x_i + \frac{\partial F_j}{\partial y_i}\delta y_i + \frac{\partial F_j}{\partial z_i}\delta z_i\right) = 0 \qquad (j = 1, 2, \ldots, m). \tag{I}$$

Therefore: every set of numbers $\delta x_1, \ldots, \delta z_n$, defining a virtual displacement of a system of points, satisfies the system of equations (I). Conversely, every set of numbers $\delta x_1, \ldots, \delta z_n$, satisfying the system of equations (I), defines a virtual displacement.

Making use of this fact, we can give the following definition of virtual displacements (equivalent to the preceding):

A virtual displacement of a system whose constraints are given by equations (12) *is one in which every displacement* $\delta x_1, \ldots, \delta z_n$, *satisfies equations* (I).

The system of equations (I) (or (15)) is a system of m equations with $3n$ unknowns $\delta x_1, \ldots, \delta z_n$.

We usually assume that equations (I) are independent of one another. Because of this we can choose arbitrarily $k = 3n - m$ unknowns from among the unknowns $\delta x_1, \ldots, \delta z_n$, and we can calculate those remaining from equations (I).

If the set of numbers $\delta x_1, \ldots, \delta z_n$, satisfies equations (I), then the set of numbers $-\delta x_1, \ldots, -\delta z_n$, obviously satisfies equations (I) also.

A virtual displacement of a system of points is said to be *reversible* if upon changing in it the senses of the displacements of all its points we again obtain a virtual displacement of the system.

We see, therefore, that *in the case of bilateral constraints the virtual displacements are reversible.*

Remark 1. The differential of the function $F_j(x_1, \ldots, z_n)$ is

$$\mathrm{d}F_j = \frac{\partial F_j}{\partial x_1}\mathrm{d}x_1 + \ldots + \frac{\partial F_j}{\partial z_n}\mathrm{d}z_n = \sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\mathrm{d}x_i + \frac{\partial F_j}{\partial y_i}\mathrm{d}y_i + \frac{\partial F_j}{\partial z_i}\mathrm{d}z_i\right).$$

We see from this that the left side of equation (I) is obtained by forming the differential of the function F_j formally and then writing $\delta x_1, \ldots, \delta z_n$, instead of $\mathrm{d}x_1, \ldots, \mathrm{d}z_n$.

Remark 2. Let $V(x_1, \ldots, x_n)$ be a given function defined in a certain region of the variables $x_1, \ldots, z_n$, and continuous together with its partial derivatives in this region. Let us choose an arbitrary set of values of the variables $x_1, \ldots, z_n$, and denote by $\delta x_1, \ldots, \delta z_n$ the arbitrary increments of these variables.

We do not denote here the increments by the symbols $\Delta x_1, \ldots, \Delta z_n$, because when the variables $x_1, \ldots, z_n$, are functions of the time t the symbols $\Delta x_1, \ldots, \Delta z_n$ usually denote the increments of these variables in the time Δt. The symbols $\delta x_1, \ldots, \delta z_n$, serve to indicate, instead, that the increments are entirely arbitrary and have nothing in common with the increments of the independent variables on which $x_1, \ldots, z_n$, depend (in this instance on the time t).

By Taylor's theorem we have:

$$V(x_1 + \delta x_1, \ldots, z_n + \delta z_n) - V(x_1, \ldots, z_n) = \\ = \frac{\partial V}{\partial x_1}\delta x_1 + \ldots + \frac{\partial V}{\partial z_n}\delta z_n + R, \tag{16}$$

where the remainder R can be written in the form

$$R = \varepsilon(|\delta x_1| + \ldots + |\delta z_n|), \tag{17}$$

where ε depends on $\delta x_1, \ldots, \delta z_n$ and tends to zero together with $\delta x_1, \ldots, \delta z_n$.

Let us put

$$\delta V = \frac{\partial V}{\partial x_1}\delta x_1 + \ldots + \frac{\partial V}{\partial z_n}\delta z_n,$$

i. e.

$$\delta V = \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right). \tag{18}$$

By (16) we have

$$V(x_1 + \delta x_1, \ldots, z_n + \delta z_n) - V(x_1, \ldots, z_n) = \delta V + R. \tag{19}$$

If the number $|\delta x_1| + \ldots + |\delta z_n|$ is sufficiently small, then $|\varepsilon|$ is also a small number, and consequently by (17) $|R|$ is evanescent as compared with $|\delta x_1| + \ldots + |\delta z_n|$. In this case, therefore, δV represents approximately the increment of the function V. We express this usually by saying that δV denotes the increment of the function V corresponding to the "infinitesimal" increments $\delta x_1, \ldots, \delta z_n$, of the variables $x_1, \ldots, z_n$, or that for "infinitesimal" increments we have

$$V(x_1 + \delta x_1, \ldots, z_n + \delta z_n) - V(x_1, \ldots, z_n) = \delta V. \tag{20}$$

The preceding statement is not altogether exact, but it is convenient. We give it because physicists use it frequently.

By (18) equations (I), p. 426, defining the virtual displacements, can be written in the form

$$\delta F_j = 0 \qquad (j = 1, 2, \ldots, m). \tag{21}$$

In a position of a system compatible with the constraints we have $F_j = 0$ $(j = 1, 2, \ldots, m)$. Hence by (21) we have $F_j + \delta F_j = 0$ $(j = 1, 2, \ldots, m)$ for the virtual displacements. In virtue of (20) we can then say that after an "inifinitesimal" virtual displacement the system is likewise in a position compatible with the constraints. This gives rise to the definition of a virtual displacement as an "infinitesimal displacement compatible with the constraints". This definition (not exact, but rather intuitive) is to be understood in the above given sense.

Example 3. A system consisting of two material points A_1, A_2, has to maintain the constant distance $A_1A_2 = r$. The coordinates x_1, y_1, z_1, and x_2, y_2, z_2, of these points consequently satisfy the equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - r^2 = 0. \tag{22}$$

The virtual displacement of the system is threfore defined by the equation

$$(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) + \\ + (z_1 - z_2)\cdot(\delta z_1 - \delta z_2) = 0. \tag{23}$$

Of the numbers δx_1, δy_1, δz_1, δx_2, δy_2, δz_2, we can hence choose five arbitrarily and determine the sixth from (23).

Let us assume, for example, that $x_1 = y_1 = z_1 = 0$, $x_2 = r$, $y_2 = z_2 = 0$. Then equation (23) assumes the form

$$-r(\delta x_1 - \delta x_2) + 0\cdot(\delta y_1 - \delta y_2) + 0\cdot(\delta z_1 - \delta z_2) = 0. \tag{24}$$

If we take $\delta x_1 = \delta x_2$, equation (24) will be satisfied for arbitrary values of δx_2, δy_1, δy_2, δz_1, δz_2. The virtual displacements of the points A_1, A_2, therefore have equal projections on the direction A_1A_2. This follows easily from the theorem given on p. 321, according to which the projections of the velocities of the points A_1, A_2, on the line joining these points are equal.

Example 4. **Unconstrained rigid body.** Let us assume that a rigid body is a rigid system of material points (p. 190).

Let us give the body an arbitrary advancing motion of velocity $\overline{\delta u}$. Since the points of the body will have this same velocity $\overline{\delta u}$, the virtual displacement of the body is obtained by assuming that the displacements of the points of the body were equal:

$$\overline{\delta s} = \overline{\delta u}. \tag{25}$$

It follows from this that *a translation of a body is a virtual displacement.*

Let us give a body an arbitrary rotation with an angular velocity $\overline{\delta\omega}$ about an axis passing through an (arbitrary) point O (Fig. 307). The velocity of an arbitrary point A of the body is equal to $\overline{\delta w} = \overline{OA} \times \overline{\delta\omega}$ (p. 46). The virtual velocity of the body is therefore obtained by giving the arbitrary point A a displacement $\overline{\delta s} = \overline{\delta w}$, i. e.

$$\overline{\delta s} = \overline{OA} \times \overline{\delta\omega}. \tag{26}$$

It follows from this that *the virtual displacement of a body is obtained by giving the points of the body displacements proportional to the velocity which they would have if the body were rotating about an arbitrary axis with an arbitrary angular velocity.*

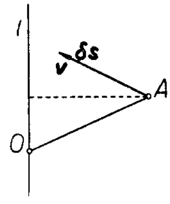

Fig. 307.

Let us note that in this case the virtual displacement is not a possible displacement.

The most general instantaneous motion of a rigid body is the composition of an advancing motion and a rotation (p. 332).

Therefore: *the most general virtual displacement of a body is the composition of two virtual displacements of which one is a translation, and in the other the displacements of the points are proportional to their velocities during a rotation of the body about an axis.*

By (25) and (26) the most general virtual displacement of a body is obtained by choosing an arbitrary point O as well as the vectors $\overline{\delta u}$, $\overline{\delta \omega}$ and giving every point A of the body a displacement

$$\overline{\delta s} = \overline{\delta u} + \overline{OA} \times \overline{\delta \omega}. \tag{27}$$

So far we have assumed that the rigid body is free. Let us now consider several cases of a constrained body.

Fixed point. If a rigid body has one fixed point, e. g. the point O, then it can only rotate about this point. The instantaneous motion of the body is consequently an instantaneous rotation about a certain axis passing through O (p. 331). The most general virtual displacement of the body is obtained by giving the points of the body displacements defined by formula (26), in which $\overline{\delta \omega}$ can be chosen arbitrarily.

Fixed axis. If a rigid body has a fixed axis, then the motion of the body can only be a rotation about this axis. Therefore in a virtual displacement of a body the points have displacements defined by formula (26), where O is an arbitrary point of the axis and $\overline{\delta \omega}$ is an arbitrary vector having the direction of the axis.

Motion of a figure in a plane. The instantaneous motion of a plane figure in its plane is either an advancing motion or a rotation about the instantaneous centre of rotation (p. 326).

In the most general case, therefore, the virtual displacement of a plane figure is either a translation or a displacement, in which the displacements of the points of the figure are proportional to the velocities of these points in a rotation about a certain point lying in the plane of the figure.

Example 5. Two material points A_1 and A_2, joined by a rigid (massless) rod of length d, are constrained to remain on the curves C_1 and C_2 lying in a horizontal plane and given by the equations:

$$f_1(x, y) = 0, \; f_2(x, y) = 0. \tag{28}$$

The following relations among the coordinates of the points A_1 and A_2 therefore hold:

$$f_1(x_1, y_1) = 0,\ f_2(x_2, y_2) = 0, \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 - d^2 = 0. \quad (29)$$

Equations (29) define the constraints of the system. The virtual displacement consequently satisfies the equations:

$$\begin{gathered}\frac{\partial f_1}{\partial x_1}\,\delta x_1 + \frac{\partial f_1}{\partial y_1}\,\delta y_1 = 0, \quad \frac{\partial f_2}{\partial x_2}\,\delta x_2 + \frac{\partial f_2}{\partial y_2}\,\delta y_2 = 0,\\ (x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0.\end{gathered} \quad (30)$$

Hence we can select one of the numbers $\delta x_1, \delta y_1, \delta x_2, \delta y_2$, arbitrarily and obtain those remaining from equations (30). Let us note that the velocities of the points A_1, A_2, are tangent to the curves C_1, C_2. The instantaneous centre of rotation O of the rod A_1A_2 is therefore the point of intersection of the normals at the points A_1 and A_2 to the curves C_1 and C_2 (cf. example 1, p. 327). Since the instantaneous motion of the rod can only be a rotation about O, the points A_1 and A_2 can only have velocities whose directions are tangent to C_1 and C_2 and whose magnitudes are proportional to OA_1 and OA_2. The virtual displacement of the system of points A_1, A_2 is therefore obtained by giving these points displacements tangent to the curves C_1, C_2 whose magnitudes are proportional to the distances OA_1, OA_2, and whose senses are as in the rotation about O.

Unilateral constraints. Let us suppose that among the relations that the coordinates of the points of a system must satisfy, there appears the unequality

$$\Phi(x_1, \ldots, z_n) \leqq 0. \quad (31)$$

Let us assume that $\Phi < 0$ in a certain position of the system. If at a certain instant the system is given an arbitrary motion compatible with all the relations except (31), then — as it is easy to see — in a small interval of time $\Phi < 0$ constantly (on account of continuity). Therefore the motion will satisfy relation (31). It follows from this that in a position in which the inequality $\Phi < 0$ holds, relation (31) does not constitute any limitation on the possible velocities and (as a consequence of this) on the virtual displacements. In determining the virtual displacements in this case, therefore, we need not take inequality (31) into account at all.

Let us now assume that the system occupies a boundary position, i. e. that the equality $\Phi = 0$ holds. At the instant t let us give the system an arbitrary motion compatible with the constraints. The function Φ will have the value $\Phi' = \Phi + \Delta\Phi$ at the time $t + \Delta t$ (where $\Delta t > 0$). Since $\Phi' \leqq 0$ and $\Phi = 0$, $\Delta\Phi \leqq 0$. Consequently $\lim\limits_{\Delta t \to 0} \Delta\Phi \,/\, \Delta t \leqq 0$, whence

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial x_1} x_1^{\cdot} + \ldots + \frac{\partial\Phi}{\partial z_n} z_n^{\cdot} \leqq 0. \tag{32}$$

The possible velocities must therefore satisfy inequality (32). If $\delta x_1, \ldots, \delta z_n$ is a virtual displacement of the system, then putting $x_1^{\cdot} = \delta x_1, \ldots, z_n^{\cdot} = \delta z_n$, we obtain a system of possible velocities. Consequently $x_1^{\cdot}, \ldots, z_n^{\cdot}$, satisfy inequality (32), whence

$$\frac{\partial\Phi}{\partial x_1} \delta x_1 + \ldots + \frac{\partial\Phi}{\partial z_n} \delta z_n \leqq 0. \tag{33}$$

Therefore, if relation (31) becomes an equality in a certain position of the system, then the virtual displacement must satisfy relation (33) and conversely: a displacement satisfying relation (33) is a virtual displacement.

If the sign „ $<$ " appears in relation (33) for a certain virtual displacement $\delta x_1, \ldots, \delta z_n$, then the displacement $-\delta x_1, \ldots, -\delta z_n$, is not a virtual displacement; hence the given virtual displacement is irreversible (p. 427). On the other hand, if the displacement $\delta x_1, \ldots, \delta z_n$ is reveresible, the sign „ $=$ " must appear in (33).

Collecting the results obtained, we can therefore say:

If the constraints of a system are defined by the relations:

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m),$$
$$\Phi_r(x_1, \ldots, z_n) \leqq 0 \qquad (r = 1, 2, \ldots, s),$$

then the virtual displacement $\delta x_1, \ldots, \delta z_n$ in a given position of the system satisfies the equations:

$$\sum_{i=1}^{n} \left(\frac{\partial F_j}{\partial x_i} \delta x_i + \frac{\partial F_j}{\partial y_i} \delta y_i + \frac{\partial F_j}{\partial z_i} \delta z_i \right) = 0 \qquad (j = 1, 2, \ldots, m) \tag{I}$$

as well as those relations from among

$$\sum_{i=1}^{n} \left(\frac{\partial \Phi_r}{\partial x_i} \delta x_i + \frac{\partial \Phi_r}{\partial y_i} \delta y_i + \frac{\partial \Phi_r}{\partial z_i} \delta z_i \right) \leqq 0 \qquad (r = 1, 2, \ldots, s) \tag{II}$$

for which the equality $\Phi_r = 0$ holds in this position of the system.

Example 6. Let us assume that a material point is constrained to remain within the sphere $x^2 + y^2 + z^2 - r^2 = 0$ or on its surface. The coordinates x, y, z, of this point must consequently satisfy the inequality

$$x^2 + y^2 + z^2 - r^2 \leqq 0. \tag{34}$$

If the point lies inside the sphere then the inequality

$$x^2 + y^2 + z^2 - r^2 < 0 \tag{35}$$

holds; in this position the point can have an arbitrary velocity and therefore every velocity is a possible velocity. It follows from this that every displacement is then a virtual displacement.

If the point is on the surface of the sphere, then

$$x^2 + y^2 + z^2 - r^2 = 0, \tag{36}$$

and possible velocities are velocities tangent to the sphere or velocities having senses towards the interior of the sphere. In that case the virtual displacements are therefore displacements whose directions are tangent to the sphere, as well as those whose directions are not tangent, but have a sense towards the interior of the sphere (Fig. 308).

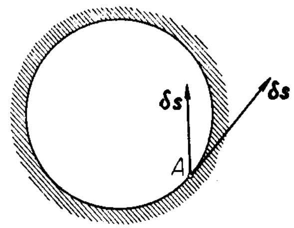

Fig. 308.

By (II) the virtual displacement $\delta x, \delta y, \delta z$, satisfies the inequality which we obtain by differentiating (34):

$$2x\,\delta x + 2y\,\delta y + 2z\,\delta z \leqq 0, \quad \text{i. e.} \quad x\,\delta x + y\,\delta y + z\,\delta z \leqq 0.$$

Example 7. A material point, tied to a string of length l attached to the origin of a coordinate system, is constrained to remain on the surface $z = x^2 + y^2$. The coordinates of the point therefore satisfy the relations:

$$z - x^2 - y^2 = 0, \quad x^2 + y^2 + z^2 - l^2 \leqq 0. \tag{37}$$

If the string is not in tension, i. e. if $x^2 + y^2 + z^2 - l^2 < 0$, then the virtual displacements satisfy only the equality

$$\delta z - 2x\,\delta x - 2y\,\delta y = 0. \tag{38}$$

It the string is in tension, then the point occupies a boundary position; hence $x^2 + y^2 + z^2 - l^2 = 0$; consequently in this case the inequality

$$x\,\delta x + y\,\delta y + z\,\delta z \leqq 0 \tag{39}$$

must hold in addition to the equality (38).

From (38) we get

$$\delta z = 2x\,\delta x + 2y\,\delta y, \tag{40}$$

whence after substituting in (39)

$$x(1 + 2z)\,\delta x + y(1 + 2z)\,\delta y \leqq 0. \tag{41}$$

Let us put

$$w = x(1 + 2z)\,\delta x + y(1 + 2z)\,\delta y. \tag{42}$$

Hence, if $y \neq 0$, then

$$\delta y = [w - x(1 + 2z)\,\delta x] \,/\, y(1 + 2z). \tag{43}$$

Therefore, for every value δx and every non-positive value of w we obtain from (40) and (43) the virtual displacement δx, δy, δz, satisfying relations (38) and (39).

§ 3. Principle of virtual work. Virtual work. Let a holonomo-scleronomic system consist of n material points $A_1, \ldots, A_n$, at which the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, are applied. Let us give the system an arbitrary virtual displacement $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$. The work of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$ on this displacement is

$$\delta' L = \mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n} = \sum_{i=1}^{n} \mathbf{P}_i \overline{\delta s_i}. \tag{I}$$

If $\delta x_1, \delta y_1, \delta z_1, \ldots, \delta x_n, \delta y_n, \delta z_n$ are the projections of the vectors $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, then

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i). \tag{I'}$$

The work defined by formulae (I) and (I′) is called the *virtual work* of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$ on the virtual displacement $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$ (having the projections $\delta x_1, \ldots, \delta z_n$).

Remark. The virtual work is denoted by $\delta' L$ (with the prime) because the symbol δL could suggest the supposition that L is a function and δL an expression defined by formula (18), p. 428.

If the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, have a potential V, then by (III), p. 211, we have $P_{i_x} = \partial V / \partial x_i$, $P_{i_y} = \partial V / \partial y_i$, $P_{i_z} = \partial V / \partial z_i$.

From (I′) we get

$$\delta' L = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i}\, \delta x_i + \frac{\partial V}{\partial y_i}\, \delta y_i + \frac{\partial V}{\partial z_i}\, \delta z_i \right), \tag{1}$$

whence by (18), p. 428,

$$\delta' L = \delta V. \tag{I''}$$

Example I. A point is constrained to the sphere $x^2 + y^2 + z^2 - r^2 = 0$. The virtual displacement is defined by the equation

$$x\, \delta x + y\, \delta y + z\, \delta z = 0.$$

If the force $\mathbf{P}$ acts on the point, then its virtual work is

$$\delta' L = P_x\, \delta x + P_y\, \delta y + P_z\, \delta z. \tag{2}$$

Let us assume that $z \neq 0$. Consequently

$$\delta z = -(x\, \delta x + y\, \delta y) / z, \tag{3}$$

whence after substituting in (2)

$$\delta' L = [(P_x z - P_z x)\, \delta x + (P_y z - P_z y)\, \delta y] \,/\, z. \tag{4}$$

The values δx, δy, in formula (4) are arbitrary.

If

$$P_x z - P_z x = 0, \quad P_y z - P_z y = 0, \tag{5}$$

then we obtain as the virtual work $\delta' L = 0$ for every virtual displacement. Conversely, if $\delta' L = 0$ constantly, then taking in formula (4) first $\delta x = 1$, $\delta y = 0$, and then $\delta x = 0$, $\delta y = 1$, we obtain equations (5) which express the fact that the direction of the force $\mathbf{P}$ passes through the origin of the coordinate system (or through the centre of the sphere), i. e. that the force is normal to the surface of the sphere.

The results obtained can be verified in the following way. Let us note that the virtual displacement at an arbitrary point of the sphere is every vector tangent to the sphere at this point (p. 422). The virtual work of the force $\mathbf{P}$ will therefore be constantly zero then, and only then, when the force $\mathbf{P}$ is perpendicular to every virtual displacement, and hence to a plane tangent to the sphere, i. e. when the direction of the force is normal to the surface of the sphere.

Principle of virtual work. Let a holonomo-scleronomic system of material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, be acted upon by forces. The forces that cause the system to maintain the constraints are called **reactions**, and the forces that act at the points of the system and are not reactions are called, in order to distinguish them from the former, *acting forces*. When the system is at rest, i. e. in equilibrium, the acting forces are said to *balance one another*.

It is obvious that a system does not have to be at rest even when the acting forces balance one another: e. g. a material point on which no forces act can move with a uniform motion.

Let us assume that a system of points is in equilibrium. At each separate point the forces acting on this point are therefore annulled by the reactions. Denoting by $\mathbf{P}_1, \ldots, \mathbf{P}_n$, the forces acting on the individual points, and by $\mathbf{R}_1, \ldots, \mathbf{R}_n$, the reactions, we hence obtain:

$$\mathbf{P}_1 + \mathbf{R}_1 = 0, \quad \mathbf{P}_2 + \mathbf{R}_2 = 0, \quad \ldots, \quad \mathbf{P}_n + \mathbf{R}_n = 0. \tag{6}$$

Let us consider an arbitrary virtual displacement $\delta s_1, \ldots, \delta s_n$. The work of the acting forces and reactions on this displacement, in view of (6), is

$$(\mathbf{P}_1 + \mathbf{R}_1)\, \overline{\delta s_1} + \ldots + (\mathbf{P}_n + \mathbf{R}_n)\, \overline{\delta s_n} = 0, \tag{7}$$

i. e.

$$(\mathbf{P}_1\, \overline{\delta s_1} + \ldots + \mathbf{P}_n\, \overline{\delta s_n}) + (\mathbf{R}_1\, \overline{\delta s_1} + \ldots + \mathbf{R}_n\, \overline{\delta s_n}) = 0. \tag{8}$$

Experience shows that if there is no friction, the work of the reactions on every virtual displacement is non-negative.

For example, if a point is constrained to remain within a certain smooth sphere, or on its surface, then, in the case when the point is on the sphere, the reaction is directed towards the centre of the sphere. The virtual displacement is either tangent to the sphere or it has a sense towards the interior (p. 433). In the first case the work of the reaction is zero, but in the second case it is positive.

Therefore, under the assumption that there is no friction, we have

$$\mathbf{R}_1\,\overline{\delta s_1} + \ldots + \mathbf{R}_n\,\overline{\delta s_n} \geqq 0. \tag{9}$$

By (8)

$$\mathbf{P}_1\,\overline{\delta s_1} + \ldots + \mathbf{P}_n\,\overline{\delta s_n} = -(\mathbf{R}_1\,\overline{\delta s_1} + \ldots + \mathbf{R}_n\,\overline{\delta s_n}); \tag{10}$$

hence from (9) we obtain

$$\mathbf{P}_1\,\overline{\delta s_1} + \ldots + \mathbf{P}_n\,\overline{\delta s_n} \leqq 0. \tag{11}$$

The expression on the left side of the inequality (11) represents, according to (I), p. 434, the virtual work of the acting forces.

Therefore: *if there is no friction and a system is in equilibrium, the work of the acting forces for every virtual displacement is either zero or a negative number.*

If the virtual displacement $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, is reversible (p. 427), then $-\overline{\delta s_1}, \ldots, -\overline{\delta s_n}$, is also a virtual displacement. In the case of equilibrium, (11) as well as

$$-\mathbf{P}_1\,\overline{\delta s_1} - \ldots - \mathbf{P}_n\,\overline{\delta s_n} \leqq 0 \tag{12}$$

hold.

From (11) and (12) it follows that

$$\mathbf{P}_1\,\overline{\delta s_1} + \ldots + \mathbf{P}_n\,\overline{\delta s_n} = 0. \tag{13}$$

Therefore: in the case of the equilibrium of a system the virtual work of the acting forces is equal to zero for every reversible virtual displacement.

In particular, if the constraints are bilateral, every virtual displacement is reversible and consequently the virtual work of the acting forces is then zero for every virtual displacement.

The condition of equilibrium obtained is a necessary condition. Experience teaches that it is also sufficient.

This condition is known as the *principle of virtual work.*

We can state it as follows:

Principle of virtual work. *If a system of n material points $A_1, \ldots, A_n$, is holonomo-scleronomic and there is no friction, then the necessary and sufficient condition for the equilibrium of the acting forces* $\mathbf{P}_1, \ldots, \mathbf{P}_n$, *is that* *for every virtual displacement* $\delta x_1, \ldots, \delta z_n$ *the virtual work of the acting forces be zero or a negative number*, i. e. *that the relation*

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i) \leqq 0 \tag{II}$$

hold.

If the constraints are bilateral, condition (II) *assumes the form*

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i) = 0. \tag{III}$$

In many cases the principle of virtual work can be proved. We accept it as a law verified by experience in all those cases in which the concept of friction is defined. In the general case it can be said that friction does not appear in a system if the principle of virtual work applies to the system.

The importance of the principle of virtual work consists in the fact that it gives the condition for the equilibrium of the acting forces without the aid of the reactions.

Example 2. Let A be a free material point. Let us denote by $\mathbf{P}$ the sum of the forces acting on A. The virtual work is $\delta' L = \mathbf{P}\, \overline{\delta s}$, where $\overline{\delta s}$ is a virtual displacement. In the case of equilibrium $\delta' L = 0$, i. e.

$$\mathbf{P}\, \overline{\delta s} = 0 \tag{14}$$

for every virtual displacement. Since the point A is free, $\overline{\delta s}$ is arbitrary. It follows from this, in view of (14), that $\mathbf{P} = 0$. For were $\mathbf{P} \neq 0$, then assuming that $\overline{\delta s}$ has the direction and sense of the force $\mathbf{P}$, we should have $\mathbf{P}\, \overline{\delta s} = |\mathbf{P}| \cdot |\overline{\delta s}| \neq 0$, contrary to (14).

We have thus verified the principle of virtual work in the case of a free point.

Example 3. A material point A, subjected to the action of the force $\mathbf{P}$, is constrained to remain on the surface S. In the position of equilibrium the virtual work is $\delta' L = \mathbf{P}\, \overline{\delta s} = 0$, where $\overline{\delta s}$ is a virtual displacement and hence an arbitrary vector lying in the plane Π tangent to the surface S at the point A (p. 422). It follows from this that $\mathbf{P} \perp \Pi$. Conversely, if $\mathbf{P} \perp \Pi$, then obviously $\delta' L = \mathbf{P}\, \overline{\delta s} = 0$, and hence the point A is in the position of equilibrium.

Let us assume now that the point A is constrained to lie on one side of the surface S. The constraints are consequently unilateral. In the case of equilibrium we therefore have $\delta' L = \mathbf{P}\, \overline{\delta s} \leqq 0$ for every virtual displacement. If $\overline{\delta s}$ lies in the tangent plane Π, then it is a reversible displacement (p. 427), and hence $\mathbf{P}\, \overline{\delta s} = 0$. It follows from this that $\mathbf{P} \perp \Pi$.

The most general virtual displacement is any vect or (whose origin is at A) directed towards that side of the surface S on which the point A lies. Since $\mathbf{P}\,\overline{\delta s} \leq 0$, $\mathbf{P}$ has a sense in the direction of the surface S (i. e. it presses the point A to the surface S). Conversely, if $\mathbf{P} \perp \Pi$ and $\mathbf{P}$ has a sense in the direction of the surface S, then, as is easily seen, $\delta' L = \mathbf{P}\,\overline{\delta s} \leq 0$ for every virtual displacement. The point is consequently in the position of equilibrium.

Example 4. The material point A, subjected to the action of the force $\mathbf{P}$, is constrained to remain on the curve C. In the position of equilibrium the virtual work for every virtual displacement δs is $\delta' L = \mathbf{P}\,\overline{\delta s} = 0$. Since $\overline{\delta s}$ is an arbitrary vector tangent to C at the point A, $\mathbf{P}$ is perpendicular to C.

Conversely, if $\mathbf{P}$ is perpendicular to C, then obviously $\mathbf{P}\,\overline{\delta s} = 0$, and hence the point is in the position of equilibrium.

Example 5. A lever AB is acted upon by weights $\mathbf{Q}_1$, $\mathbf{Q}_2$, suspended from the points A, B, and as the weight $\mathbf{Q}$ acting at the centre S of its mass. The acting forces lie in a vertical plane perpendicular to the axis of rotation at the point O, while $OS \perp AB$. Determine in the position of equilibrium the angle φ which OS makes with the vertical (cf. example 1, p. 274).

Let us denote by $\overline{\delta s_1}$, $\overline{\delta s_2}$, $\overline{\delta s}$, the virtual displacements of the points of application A, B, S. The lever can only rotate about its axis. The possible velocities, and — as a consequence — the virtual displacements of the points A, B, S are perpendicular to OA, OB, OS, (Fig. 309). Denoting by $\delta\omega$ an arbitrary angular velocity, we consequently have:

$$|\overline{\delta s_1}| = OA\ \delta\omega, \quad |\overline{\delta s_2}| = OB\ \delta\omega, \quad |\overline{\delta s}| = OS\ \delta\omega. \tag{15}$$

In the position of equilibrium the virtual work is zero, e. i. $\mathbf{Q}_1\,\overline{\delta s_1} + \mathbf{Q}_2\,\overline{\delta s_2} + \mathbf{Q}\,\overline{\delta s} = 0$. Calculating the scalar products and denoting the absolute values of the forces by Q_1, Q_2, Q, we obtain by (15) $(Q_1 \cdot OA \cos\varphi + Q \cdot OS \sin\varphi - Q_2 \cdot OB \cos\varphi)\ \delta\omega = 0$, whence after dividing by $\delta\omega$

$$\tan\varphi = (Q_2 \cdot OB - Q_1 \cdot OA)\ /\ Q \cdot OS. \tag{16}$$

Formula (16) was obtained before in another way (cf. formula (6), p. 275).

Example 6. On an inclined plane making an angle α with the horizontal there lies a heavy point A which remains in equilibrium under action of a force $\mathbf{P}$ having a horizontal direction (Fig. 310). Determine the force $\mathbf{P}$ under the assumption that there is no friction.

Let us denote by $\mathbf{Q}$ the weight of the body and by $\overline{\delta s}$ an arbitrary virtual displacement. The virtual work is $\delta' L = \mathbf{Q}\,\overline{\delta s} + \mathbf{P}\,\overline{\delta s}$. In order that equilibrium occur, we must have for every virtual displacement $\delta' L \leqq 0$, i. e.

$$\mathbf{Q}\,\overline{\delta s} + \mathbf{P}\,\overline{\delta s} \leqq 0. \tag{17}$$

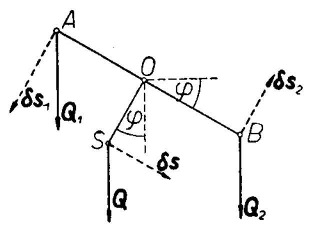

Fig. 309.

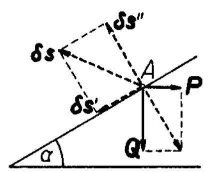

Fig. 310.

Let us first consider the virtual displacement $\overline{\delta s'}$ having the direction of the inclined plane. Under this assumption $\overline{\delta s'}$ is a reversible displacement; consequently (17) assumes the form of the equality

$$\mathbf{Q}\,\overline{\delta s'} + \mathbf{P}\,\overline{\delta s'} = 0. \tag{18}$$

Let Π be a vertical plane passing through A and perpendicular to the inclined plane, and let $\overline{\delta s'} \perp \Pi$. Therefore $\mathbf{Q}\,\overline{\delta s'} = 0$, whence by (2) $P \cdot \overline{\delta s'} = 0$, i. e. $\mathbf{P} \perp \overline{\delta s'}$. Hence $\mathbf{P}$ lies in the plane Π.

Let us now assume that $\overline{\delta s'}$ lies on the inclined plane and in the plane Π (Fig. 310); giving the displacement $\overline{\delta s'}$ a downward sense and putting $Q = |\mathbf{Q}|$, $P = |\mathbf{P}|$, we obtain from (18)

$$Q|\overline{\delta s'}| \sin\alpha \pm P|\overline{\delta s'}| \cos\alpha = 0. \tag{19}$$

The sign „$\pm$" depends on the sense of the force $\mathbf{P}$. From the equality (19) it follows that it is necessary to take the sign „—". Hence the force $\mathbf{P}$ must press the point to the plane. Using the sign „—" we obtain from (19)

$$P = Q \tan\alpha. \tag{20}$$

We have thus determined the direction, sense, and magnitude, of the force $\mathbf{P}$ under the assumption of equilibrium. From (20) it follows easily that the sum $\mathbf{Q} + \mathbf{P}$ is perpendicular to the inclined plane.

In order to show that equilibrium really occurs, it is necessary to prove that condition (17) holds for every virtual displacement. In order to demonstrate this, let us resolve the arbitrary displacement $\overline{\delta s}$ into two displacements: $\overline{\delta s''}$ perpendicular to the inclined plane and $\overline{\delta s'}$ lying on the inclined plane. The work of the forces $\mathbf{P}$ and $\mathbf{Q}$ on the displacement $\overline{\delta s'}$ is zero, because $\mathbf{P} + \mathbf{Q} \perp \overline{\delta s'}$. The displacement $\overline{\delta s''}$ and the sum

$\mathbf{P} + \mathbf{Q}$ have the same direction, but opposite senses; hence the work of the forces $\mathbf{P}$ and $\mathbf{Q}$ on $\overline{\delta s''}$ is negative. It follows from this that the work of the forces $\mathbf{P}$ and $\mathbf{Q}$ on the displacement $\overline{\delta s}$ is negative. Relation (17) therefore holds for every virtual displacement. Consequently the forces $\mathbf{Q}$ and $\mathbf{P}$ balance each other.

Example 7. Rigid body. By appealing to the principle of virtual work we shall now derive the conditions for the equilibrium of forces acting on a rigid body.

Unconstrained body. Let the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, having origins $A_1, \ldots, A_n$, act on a free rigid body. Let us give the body an arbitrary virtual displacement and denote by $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, the displacements of the points $A_1, \ldots, A_n$. Hence by (I), p. 434, the virtual displacement is

$$\delta' L = \mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n}. \tag{21}$$

In example 4, p. 429, we considered the virtual displacement of a rigid body.

Let the virtual displacement of the body be a translation (p. 429); the displacements of the points are therefore equal $\overline{\delta u}$. By (21) we have $\delta' L = \mathbf{P}_1 \overline{\delta u} + \ldots + \mathbf{P}_n \overline{\delta u} = (\mathbf{P}_1 + \ldots + \mathbf{P}_n) \overline{\delta u}$. Putting $\mathbf{P} = \mathbf{P}_1 + \ldots + \mathbf{P}_n$, we obtain

$$\delta' L = \mathbf{P} \overline{\delta u}. \tag{22}$$

Let O be an arbitrary point of the body and l an axis passing through O. Let us give the body a virtual displacement, in which displacements of the points are proportional to the velocities during a rotation of the body about the axis l with an angular velocity $\overline{\delta\omega}$. By (26), p. 429, we obtain

$$\overline{\delta s_i} = \overline{OA_i} \times \overline{\delta\omega} \qquad (i = 1, 2, \ldots, n),$$

whence by (21) $\delta' L = \mathbf{P}_1(\overline{OA_1} \times \overline{\delta\omega}) + \ldots + \mathbf{P}_n(\overline{OA_n} \times \overline{\delta\omega})$. Since $\boldsymbol{a}(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}(\boldsymbol{a} \times \boldsymbol{b})$ (formula (II), p. 13),

$$\delta' L = \overline{\delta\omega}(\mathbf{P}_1 \times \overline{OA_1}) + \ldots + \overline{\delta\omega}(\mathbf{P}_n \times \overline{OA_n}).$$

But $\mathbf{P}_1 \times \overline{OA_1} = \mathrm{Mom}_O \mathbf{P}_1$, etc. Consequently

$$\delta' L = \overline{\delta\omega}(\mathrm{Mom}_O \mathbf{P}_1 + \ldots + \mathrm{Mom}_O \mathbf{P}_n) = \overline{\delta\omega} \cdot \mathbf{M}, \tag{23}$$

where $\mathbf{M}$ is the total moment of the forces with respect to O.

Let us denote by $\delta\omega$ the component of $\overline{\delta\omega}$ with respect to the axis l, and by φ the angle which $\mathbf{M}$ makes with the axis l. Then $\overline{\delta\omega} \cdot \mathbf{M} = \delta\omega \cdot |\mathbf{M}| \cos\varphi$. Since the moment of the acting forces with respect to the axis l is $M_l = |\mathbf{M}| \cos\varphi$, by (23)

$$\delta' L = M_l \, \delta\omega. \tag{24}$$

The most general virtual displacement of a rigid body is a composition of the displacements defined by formulae (25) and (26), p. 429. Therefore for the most general virtual displacement

$$\delta' L = \mathbf{P} \cdot \overline{\delta u} + M_l \, \delta\omega, \tag{25}$$

where $\mathbf{P}$ denotes the sum of all the acting forces, M_l the moment with respect to an arbitrary axis l, $\overline{\delta u}$ an arbitrary vector, and $\delta\omega$ an arbitrary number.

We now proceed to determine the conditions for equilibrium. Let us assume that the system of acting forces is in equilibrium. Hence the virtual work $\delta' L$ on every virtual displacement is zero. Giving the body an arbitrary displacement $\overline{\delta u}$, we have by (22) $\mathbf{P}\,\overline{\delta u} = 0$, from which it follows that

$$\mathbf{P} = 0, \tag{26}$$

for were $\mathbf{P} \neq 0$, then choosing $\overline{\delta u}$ in the direction of $\mathbf{P}$, we should have $\mathbf{P}\,\overline{\delta u} \neq 0$.

Let us now select an arbitrary point O and an axis l passing through O. By (24) $M_l \, \delta\omega = 0$ for every $\delta\omega$; hence $M_l = 0$. Since l is an arbitrary axis passing through O, the total moment with respect to O is

$$\mathbf{M} = 0. \tag{27}$$

In this way we have proved that the equalities (26) and (27) are *necessary* conditions for the equilibrium of the acting forces. We shall now show that they are likewise *sufficient* conditions.

For if the equalities (26) and (27) hold, then the virtual work given in the most general case by formula (25) is obviously zero.

We have therefore obtained the conditions of equilibrium for a free rigid body, which were derived in another way on p. 244.

We shall now consider several cases of equilibrium of a constrained body.

Body having a fixed point. Let a body have the point O fixed. The instantaneous motion of the body can only be a rotation about an axis passing through O. Consequently the virtual work is expressed by formula (24).

If the acting forces balance one another, then $\delta' L = 0$, whence by (24) $M_l \, \delta\omega = 0$. Since $\delta\omega$ is arbitrary, $M_l = 0$, where l is an arbitrary axis passing through O. It follows from this that the total moment with respect to O of the forces acting on the body is $\mathbf{M} = 0$.

Conversely, if $\mathbf{M} = 0$, then $M_l = 0$ with respect to every axis l passing through O. Hence by (24) $\delta' L = 0$ constantly.

Therefore: a system of forces acting on a body having one fixed point O is in equilibrium then, and only then, when the total moment of the forces with respect to O is zero.

This condition was obtained before in another way (p. 271).

Plane motion of a body (p. 272). Let Π be the directional plane of a body in plane motion. The instantaneous motion of the body is either an advancing motion with a velocity parallel to Π, or a rotation about an axis l perpendicular to Π. Consequently the virtual work is expressed by formula (22), where $\overline{\delta u} \parallel \Pi$, or by formula (24), where $l \perp \Pi$. In the case of equilibrium we therefore obtain $\mathbf{P}\,\overline{\delta u} = 0$ for $\overline{\delta u} \parallel \Pi$, and $M_l\,\delta\omega = 0$ for $l \perp \Pi$. It follows from this that

$$\mathbf{P} \perp \Pi \text{ and } M_l = 0 \text{ for } l \perp \Pi. \tag{28}$$

It is easy to prove that the condition obtained is equivalent to the condition that the projections of the forces on the directional plane Π form a system equipollent to zero. If condition (28) holds, then from (22) and (24) it follows that the virtual work is zero. Condition (28) is therefore necessary and sufficient for the equilibrium of the acting forces.

Body having a fixed axis. Let us assume that a body has a fixed axis l. In this case the body can only rotate about the axis l. The virtual work is therefore expressed by formula (24). Hence by (24) the necessary and sufficient condition for the equilibrium of the acting forces is that $\delta' L = M_l\,\delta\omega = 0$, whence $M_l = 0$ (for $\delta\omega$ is arbitrary).

We obtained this condition before on p. 272.

Body having a fixed axis of twist. Let us assume that a body can only rotate about a certain axis l as well as to move along it, which is the case e. g. with a sphere strung on a straight rigid rod. Hence the instantaneous motion of the body is the composition of an advancing motion whose velocity has the direction of the axis l and a rotation about this axis, and consequently the motion is a twist about the axis l. In the most general case the virtual work is therefore defined by formula (25), in which $\overline{\delta u}$ has the direction of the axis l and $\delta\omega$ is arbitrary.

If equilibrium occurs, then by (25)

$$\delta' L = \mathbf{P}\,\overline{\delta u} + M_l\,\delta\omega = 0. \tag{29}$$

Assuming $\overline{\delta u} = 0$ and $\delta\omega \neq 0$, we get $M_l = 0$; and if we assume $\delta\omega = 0$, we obtain from (29) $\mathbf{P}\,\overline{\delta u} = 0$. Since $\overline{\delta u}$ has the direction of the axis l, $\mathbf{P} \perp l$.

Conversely, if $M_l = 0$ and $\mathbf{P} \perp l$, then by (29) obviously $\delta' L = 0$.

Therefore: *a necessary and sufficient condition for the equilibrium of a body which can rotate about a fixed axis and slide along it, is that the sum of the forces be perpendicular to the axis and the moment of the forces with respect to the axis be zero.*

Screw. A rigid body which can only move so that a certain helix in the body slides along itself is called a *screw*.

The axis of the screw can only slide along itself, and consequently the velocities of the points on the axis have the direction of the axis. It follows from this (p. 334) that the instantaneous motion is an instantaneous twist about the axis of the screw.

Denoting by $\boldsymbol{u}$ the velocity of the instantaneous advancing motion, by $\boldsymbol{\omega}$ the instantaneous angular velocity, and by h the lead of the screw, we have by (15), p. 337,

$$|\boldsymbol{u}| \,/\, |\boldsymbol{\omega}| = h \,/\, 2\pi. \tag{30}$$

Since $\boldsymbol{u}$ and $\boldsymbol{\omega}$ have the direction of the axis l of the screw, denoting by u and ω the components with respect to the axis l of the vectors $\boldsymbol{u}$ and $\boldsymbol{\omega}$, we obtain from (30)

$$u = \varepsilon h \omega \,/\, 2\pi, \tag{31}$$

where $\varepsilon = +1$, if the screw is left-handed, and $\varepsilon = -1$, if it is right-handed.

The virtual work is expressed by formula (25), in which $\delta\omega$ is arbitrary, and $\overline{\delta u}$ has the direction of the axis l of the screw, while by (31)

$$\delta u = \varepsilon h \, \delta\omega \,/\, 2\pi, \tag{32}$$

where δu is the component of $\overline{\delta u}$ with respect to the axis l.

Denoting by P_l the projection of $\boldsymbol{P}$ on the axis l, we have $\boldsymbol{P}\,\overline{\delta u} = P_l\,\delta u$. Hence by (25) and (32)

$$\delta' L = (\varepsilon P_l h \,/\, 2\pi + M_l)\,\delta\omega. \tag{33}$$

From the principle of virtual work it follows by (33) that *a necessary and sufficient condition for the equilibrium of the forces acting on a screw is that the forces satisfy the equation*

$$M_l \,/\, P_l = -\,\varepsilon h \,/\, 2\pi. \tag{34}$$

Example 8. **Determination of stresses in the bars of a frame. Kinematical method.** A certain method of determining the stresses in the bars of a frame rests on the principle of virtual work. First we shall illustrate this method by means of an example.

Forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, act at the joints of a plane frame and are in equilibrium. In order to determine the stress in the bar BD, for example, we remove this bar. The remaining system of bars will again be in equilibrium, if at the points B and D we apply the forces $\mathbf{S}$ and $-\mathbf{S}$, equal to the stresses at these two points in the bar removed (Fig. 311).

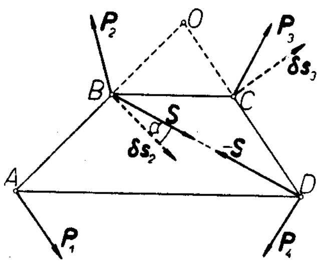

Fig. 311.

Let us give the system of bars AB, BC, CD, DA, an arbitrary virtual displacement and denote by $\delta s_1, \ldots, \delta s_4$, the displacements of the points $A, \ldots, D$. The virtual work will therefore be zero, i. e.

$$\mathbf{P}_1 \overline{\delta s_1} + \mathbf{P}_2 \overline{\delta s_2} + \mathbf{P}_3 \overline{\delta s_3} + \\ + \mathbf{P}_4 \overline{\delta s_4} + \mathbf{S} \overline{\delta s_2} - \mathbf{S} \overline{\delta s_4} = 0,$$

whence

$$\mathbf{S}(\overline{\delta s_2} - \overline{\delta s_4}) = -(\mathbf{P}_1 \overline{\delta s_1} + \mathbf{P}_2 \overline{\delta s_2} + \mathbf{P}_3 \overline{\delta s_3} + \mathbf{P}_4 \overline{\delta s_4}). \tag{35}$$

Let us put $S = \pm |\mathbf{S}|$, where the sign depends on whether the stress $\mathbf{S}$ is a tension or a compression, and let us denote by δr the projection of the difference $\overline{\delta s_2} - \overline{\delta s_4}$ on the direction of $\overline{BD}$. With these notations the left side of the equality (35) is $S\,\delta r$. Therefore, if we choose the virtual displacement in such a way that $\delta r \neq 0$, then we obtain from (35)

$$S = -(\mathbf{P}_1 \overline{\delta s_1} + \mathbf{P}_2 \overline{\delta s_2} + \mathbf{P}_3 \overline{\delta s_3} + \mathbf{P}_4 \overline{\delta s_4}) \,/\, \delta r. \tag{36}$$

The sought for virtual displacement is obtained by assuming that the points A and D are fixed; consequently:

$$\overline{\delta s_1} = 0, \quad \overline{\delta s_4} = 0. \tag{37}$$

The instantaneous motion of the bar BD (under the assumption that A and D are fixed) is an instantaneous rotation about the centre O, which is the point of intersection of the lines AB and DC (cf. example 4, p. 328).

The displacements $\overline{\delta s_2}$ and $\overline{\delta s_3}$ are proportional to the velocities of the points B and C during a rotation of the rod BC about O. Consequently $\overline{\delta s_2}$ and $\overline{\delta s_3}$ are perpendicular to AB and DC, respectively, and

$$|\overline{\delta s_3}| \,/\, |\overline{\delta s_2}| = OC \,/\, OB. \tag{38}$$

Let P_2' and P_3' denote the projections of the forces $\mathbf{P}_2$, $\mathbf{P}_3$, on the directions of $\overline{\delta s_2}$, $\overline{\delta s_3}$, and α the angle between $\overline{\delta s_2}$ and $\overline{BD}$. Consequently:

$$\mathbf{P}_2 \overline{\delta s_2} = P_2'|\overline{\delta s_2}|, \quad \mathbf{P}_3 \overline{\delta s_3} = P_3'|\overline{\delta s_3}|, \quad \delta r = |\overline{\delta s_2}| \cos\alpha. \tag{39}$$

From (36) we obtain by (37)—(39)

$$S = -(P_2' \cdot OB + P_3' \cdot OC) \,/\, OB \cos\alpha. \tag{40}$$

Let us now proceed the general case of a frame with joints $A_1, \ldots, A_n$, at which the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, act and are in equilibrium. In order to determine the stresses in the bar of the frame connecting the joints A_r and A_s, we remove the bar A_rA_s and apply to the joints A_r and A_s the forces $\mathbf{S}$ and $-\mathbf{S}$, equal to the stresses at A_r and A_s in the bar removed. Since the system of remaining bars is in equilibrium, the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n, \mathbf{S}, -\mathbf{S}$, balance one another. The virtual work is therefore zero. Denoting by $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, the virtual displacements of the joints $A_1, \ldots, A_n$ (under the assumption that the bar A_rA_s was removed), we obtain

$$\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n} + \mathbf{S}\, \overline{\delta s_r} - \mathbf{S}\, \overline{\delta s_s} = 0. \tag{41}$$

Hence, putting $S = \pm |\mathbf{S}|$ (where the sign depends on whether S is a tension or a compression) and denoting by δr the projection of $\overline{\delta s_r} - \overline{\delta s_s}$ on the direction of $\overline{A_rA_s}$, we obtain from (41)

$$S\, \delta r = -(\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n}). \tag{42}$$

If the virtual displacements can be so chosen that $\delta r \neq 0$ (i.e. so that $\overline{\delta s_r} \neq \overline{\delta s_s}$ and the difference $\overline{\delta s_r} - \overline{\delta s_s}$ is not perpendicular to A_rA_s), then from (42) we shall be able to determine S.

Now, it is possible to show that if a frame is statically determinate (p. 297), then the virtual displacement having the required properties always exists.

Let us denote by $x_1, y_1, \ldots, x_n, y_n$, the coordinates of the joints $A_1, \ldots, A_n$, and by d_{ij} the lengths of the bars A_iA_j. Consequently

$$(x_i - x_j)^2 + (y_i - y_j)^2 - d_{ij}^2 = 0. \tag{43}$$

Let $\delta x_i, \delta y_i$ be the projections of the virtual displacement of the point A_i.

Then by (43)

$$(x_i - x_j)(\delta x_i - \delta x_j) + (y_i - y_j)(\delta y_i - \delta y_j) = 0. \tag{44}$$

If the bar A_rA_s is removed, then the virtual displacements of the joints are defined by equations (44) (among which the equation corresponding to the bar A_rA_s does not appear); from these we can calculate the displacements.

The given method of determining the stresses in the bars of a frame is known as the *kinematical method.*

The kinematical method can be applied to plane frames as well as to space frames.

There also exist graphical methods of determining the possible velocities (and, as a consequence, the virtual displacements $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$) of the joints of the frame by means of the so-called *diagram of velocities* (*virtual displacements*).

§ 4. Determination of the position of equilibrium in a force field. One of the principal problems which we shall encounter in the investigation of the equilibrium of a system of material points is the determination of the position of equilibrium of the system in a given force field.

We shall here give the solution of this problem for bilateral constraints. For unilateral constraints the solution of the problem is considerably more complex and we shall therefore confine ourselves to an example (p. 450, example 2).

Let the constraints of the system of points $A_1, \ldots, A_n$, be defined by the equations

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m). \qquad (1)$$

Let us assume that the system is in a force field, i. e. that the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, acting on the points $A_1, \ldots, A_n$, are functions of the variables $x_1, \ldots, z_n$. Therefore:

$$P_{i_x} = \Phi_i(x_1, \ldots, z_n), \; P_{i_y} = \Psi_i(x_1, \ldots, z_n), \; P_{i_z} = X_i(x_1, \ldots, z_n). \qquad (2)$$

The virtual displacements satisfy the equations (I), p. 426:

$$\sum_{i=1}^{n} \left(\frac{\partial F_j}{\partial x_i} \delta x_i + \frac{\partial F_j}{\partial y_i} \delta y_i + \frac{\partial F_j}{\partial z_i} \delta z_i \right) = 0 \qquad (j = 1, 2, \ldots, m). \qquad (3)$$

In the position of equilibrium we have in virtue of the principle of virtual work

$$\sum_{i=1}^{n} (P_{i_x} \delta x_i + P_{i_y} \delta y_i + P_{i_z} \delta z_i) = 0. \qquad (4)$$

Since (3) consists of m equations, we can determine m of the n unknowns $\delta x_1, \ldots, \delta z_n$, giving the remaining $k = 3n - m$ unknowns the arbitrarily chosen values $\delta h_1, \ldots, \delta h_k$. Determining the unknowns $\delta x_1, \ldots, \delta z_n$, from equations (3) in terms of $\delta h_1, \ldots, \delta h_k$, and substituting in (4), we obtain after simplifying the equation

$$a_1 \delta h_1 + a_2 \delta h_2 + \ldots + a_k \delta h_k = 0, \qquad (5)$$

where $a_1, \ldots, a_k$, are certain numbers depending on the position of the system, i. e. on the coordinates $x_1, \ldots, z_n$.

In a position of equilibrium equality (5) must hold for every set of numbers $\delta h_1, \ldots, \delta h_k$. Assuming $\delta h_1 = 1, \delta h_2 = 0, \ldots, \delta h_k = 0$, we get $a_1 = 0$. Proceeding similarly, we obtain:

$$a_1 = 0, \quad a_2 = 0, \quad \ldots, \quad a_k = 0. \qquad (6)$$

Conversely, if the equalities (6) are satisfied in a certain position of the system, then obviously equation (5) holds and consequently the given position of the system is a position of equilibrium. Equalities (6) therefore define the position of equilibrium of the system.

From equations (6) and (1), whose total number is $k + m = 3n$, we can determine in general $3n$ unknown coordinates $x_1, \ldots, z_n$, corresponding to the position of equilibrium of the system.

Lagrange's multipliers. We shall give still another method of determining the positions of equilibrium of a system, called the *method of Lagrange's multipliers.*

Let us denote the left sides of equations (3) by $W_1, \ldots, W_m$, and the left side of equation (4) by W.

Regarding $\delta x_1, \ldots, \delta z_n$, as unknowns, we can say that in the position of equilibrium every solution of the equations $W_1 = 0, \ldots, W_m = 0$ satisfies the equation $W = 0$. Hence, by a well-known theorem from the theory of linear equations it follows that W can be represented as a linear combination of $W_1, \ldots, W_m$, i. e. that there exist numbers $a_1, \ldots, a_m$, such that for arbitrary $\delta x_1, \ldots, \delta z_n$, we have the identity

$$W = a_1 W_1 + \ldots + a_m W_m.$$

Putting $\lambda_1 = -a_1, \ldots, \lambda_m = -a_m$, we can write the above identity in the form

$$W + \lambda_1 W_1 + \ldots + \lambda_m W_m = 0 \quad \text{or} \quad W + \sum_{j=1}^{m} \lambda_j W_j = 0. \tag{7}$$

Writing the left sides of equations (3) and (4) instead of $W_1, \ldots, W_m$, and W, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} (P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i) + \\ &+ \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} \left(\frac{\partial F_j}{\partial x_i} \delta x_i + \frac{\partial F_j}{\partial y_i} \delta y_i + \frac{\partial F_j}{\partial z_i} \delta z_i \right) = 0. \end{aligned} \tag{8}$$

Arranging the left side of equation (8) according to $\delta x_1, \ldots, \delta z_n$, we get

$$\begin{aligned} \sum_{i=1}^{n} \Bigg[&\left(P_{i_x} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i} \right) \delta x_i + \left(P_{i_y} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i} \right) \delta y_i + \\ &+ \left(P_{i_z} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} \right) \delta z_i \Bigg] = 0. \end{aligned} \tag{9}$$

Equality (7), and hence also (9), holds for arbitrary $\delta x_1, \ldots, \delta z_n$; consequently the coefficients of $\delta x_1, \ldots, \delta z_n$, must be equal to zero:

$$P_{i_x} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i} = 0, \quad P_{i_y} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i} = 0, \quad P_{i_z} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} = 0 \quad \text{(I)}$$
$$(i = 1, 2, \ldots, n).$$

We have thus proved that in a position of equilibrium it is possible to choose numbers $\lambda_1, \ldots, \lambda_m$, such that equations (I) hold.

Conversely, if in a certain position of a system it is possible to choose numbers $\lambda_1, \ldots, \lambda_m$, satisfying equations (I), then equality (9) holds, and hence also (8), i. e. (7). Since, in virtue of (3), for virtual displacements $W_1 = 0, \ldots, W_m = 0$, from (7) we get $W = 0$, i. e. (4). The given position is consequently a position of equilibrium.

Therefore: *a necessary and sufficient condition for the equilibrium of forces in a certain position of a system is that there exist a set of numbers $\lambda_1, \ldots, \lambda_m$, satisfying equations* (I).

From equations (1) and (I), whose total number is $3n + m$, we can determine in general $3n + m$ unknowns, i. e. $\lambda_1, \ldots, \lambda_m$, and as the coordinates $x_1, \ldots, z_n$, defining the position of equilibrium.

The numbers $\lambda_1, \ldots, \lambda_m$, are called *Lagrange's multipliers*.

Remark. Denoting by $\mathbf{R}_1, \ldots, \mathbf{R}_n$, the forces of reaction in the position of equilibrium, we obviously have $\mathbf{P}_i + \mathbf{R}_i = 0$, i. e.

$$P_{i_x} + R_{i_x} = 0, \; P_{i_y} + R_{i_y} = 0, \; P_{i_z} + R_{i_z} = 0 \qquad (i = 1, 2, \ldots n).$$

Comparing with (I), we get:

$$R_{i_x} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i}, \; R_{i_y} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i}, \; R_{i_z} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} \; (i = 1, 2, \ldots, n). \quad \text{(II)}$$

Example 1. Two heavy material points A_1, A_2, of masses m_1, m_2, are connected by a rigid rod of length d (massless) and are constrained to remain on two lines l_1 and l_2. The line l_1 is vertical and the line l_2 cuts l_1 and makes with it an angle $\varphi = 45°$ (Fig. 312). Determine the position of equilibrium, assuming that there is no friction.

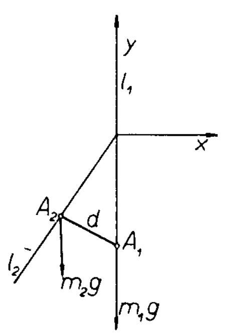

Fig. 312.

Let us choose the point of intersection of the lines l_1, l_2, as the origin of the coordinate system (x, y), taking the plane in which these lines lie as the xy-plane and the line l_1 as the y-axis (with an upward sense).

The equations of the lines l_1 and l_2 are $x = 0$ and $y = x$. Consequently the coordinates x_1, y_1, and x_2, y_2, satisfy the equations:

$$x_1 = 0, \quad y_2 - x_2 = 0. \tag{10}$$

Since $A_1A_2 = d$,

$$x_2^2 + (y_1 - y_2)^2 - d^2 = 0. \tag{11}$$

Equations (10) and (11) define the constraints of the system.

The virtual displacements satisfy the equations which we obtain from (10) and (11):

$$\delta x_1 = 0, \ \delta y_2 - \delta x_2 = 0, \ x_2 \, \delta x_2 + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0. \tag{12}$$

The acting forces are the weights of the points. The projections of the weights on the axes of the coordinate system are respectively $0, -m_1 g$, and $0, -m_2 g$. The virtual work of the acting forces is equal to $\delta' L = -m_1 g \, \delta y_1 - m_2 g \, \delta y_2$. In the position of equilibrium $\delta' L = 0$, and hence

$$m_1 \, \delta y_1 + m_2 \, \delta y_2 = 0. \tag{13}$$

Let us assume that $y_1 - y_2 \neq 0$. Selecting δy_2 arbitrarily we get from (12):

$$\delta x_1 = 0, \ \delta x_2 = \delta y_2, \ \delta y_1 = (y_1 - y_2 - x_2) \, \delta y_2 \, / \, (y_1 - y_2). \tag{14}$$

Substituting in (13) we obtain after getting rid of the denominator

$$[m_1(y_1 - y_2 - x_2) + m_2(y_1 - y_2)] \, \delta y_2 = 0. \tag{15}$$

Since δy_2 is arbitrary, equality (15) will hold only in the case when

$$m_1(y_1 - y_2 - x_2) + m_2(y_1 - y_2) = 0. \tag{16}$$

Solving the system of equations (10), (11), (16), we obtain the coordinates:

$$x_1 = 0, \ y_1 = -(2m_1 + m_2) \, d \, / \, a, \ x_2 = -(m_1 + m_2) \, d \, / \, a = y_2,$$

in the position of equilibrium, where $a = \sqrt{(m_1 m_2)^2 + m_1^2}$.

Let us assume now that $y_1 - y_2 = 0$. By (11) we have $x_2^2 - d^2 = 0$, whence $x_2 \neq 0$. In view of this the last one of the equations (12) gives $\delta x_2 = 0$, and hence the second one of the equations (12) gives $\delta y_2 = 0$. Consequently by (12) the virtual displacement is the displacement:

$$\delta x_1 = 0, \quad \delta x_2 = 0, \quad \delta y_2 = 0, \quad \delta y_1 \ \textit{arbitrary}.$$

The condition of equilibrium (13) will therefore assume the form $m_1 \, \delta y_1 = 0$. However, this equality is not satisfied, because δy_1 is arbitrary.

Therefore: the position for which $y_1 - y_2 = 0$ is not a position of equilibrium.

Example 2. A heavy point of mass m, subjected to the action of the force **P**, is constrained to remain on the surface of the sphere

$$x^2 + y^2 + z^2 - r^2 = 0. \tag{17}$$

We assume that the z-axis has a vertical direction and an upward sense. Determine the position of equilibrium, assuming that friction does not appear.

The virtual displacement $\delta x, \delta y, \delta z$, satisfies the equation which we get by differentiating (17):

$$x\,\delta x + y\,\delta y + z\,\delta z = 0. \tag{18}$$

The virtual work of the force **P** is $P_x\,\delta x + P_y\,\delta y + P_z\,\delta z$, and that of the force of gravity $-mg\,\delta z$. In the position of equilibrium we consequently have

$$\delta' L = P_x\,\delta x + P_y\,\delta y + (P_z - mg)\,\delta z = 0. \tag{19}$$

Applying the method of Lagrange's multipliers (formula (I), p. 448) and replacing 2λ by λ we obtain the following equations:

$$P_x + \lambda x = 0, \quad P_y + \lambda y = 0, \quad P_z - mg + \lambda z = 0. \tag{20}$$

From equations (17) and (20) we can determine λ as well as the coordinates x, y, z, of the position of equilibrium.

Calculating x, y, z, from equations (20) and substituting in (17), we get:

$$\lambda = \pm \sqrt{P_x^2 + P_y^2 + (P_z - mg)^2}\,/\,r. \tag{21}$$

Knowing λ we obtain from (20):

$$x = -P_x\,/\,\lambda, \quad y = -P_y\,/\,\lambda, \quad z = -(P_z - mg)\,/\,\lambda. \tag{22}$$

Since we have obtained two values (21) for λ, there will exist two positions of equilibrium.

Let us now assume that the point is constrained to remain within the sphere (17) or on its surface. The constraints are therefore unilateral and the coordinates of the point must satisfy the relation

$$x^2 + y^2 + z^2 - r^2 \leqq 0. \tag{23}$$

In the position of equilibrium on the surface of the sphere the virtual displacements are defined by the inequality

$$x\,\delta x + y\,\delta y + z\,\delta z \leqq 0. \tag{24}$$

The virtual work consequently satisfies the inequality

$$P_x\,\delta x + P_y\,\delta y + (P_z - mg)\,\delta z \leqq 0. \tag{25}$$

For reversible virtual displacements, i. e. those satisfying equation (18), the virtual work is zero and hence equality (19) holds. It follows from this that the position of equilibrium on the surface of the sphere can only be one of the positions given by formulae (21) and (22). In order to prove which one of them is a position of equilibrium, it is necessary to examine for which one of them the relation (24) implies (25).

Assuming that the virtual displacement satisfies condition (24) we obtain by (22) after substituting in (24)

$$-[P_x\,\delta x + P_y\,\delta y + (P_z - mg)\,\delta z]\,/\,\lambda \leqq 0. \tag{26}$$

We see from this that formula (25) will be satisfied only then when $-1\,/\,\lambda > 0$, i. e. when $\lambda < 0$.

Therefore formulae (21) and (22) define the position of equilibrium; in formula (21) it is necessary to choose the sign „—".

§ 5. Lagrange's generalized coordinates. Parameters of a system. The position of a system of points or of a rigid body is defined by means of certain numbers. These numbers can be, in particular, the coordinates of the points with respect to a certain rectangular coordinate system; however, in many cases they can have another meaning.

For example, the coordinates of a point in a plane can be given by means of the rectangular coordinates x, y, as well as by the polar coordinates r, φ, etc. In particular, the position of a system consisting of two points A_1, A_2, whose distance d is constant and which are constrained to lie in the xy-plane, can be defined by giving either the coordinates x_1, y_1 and x_2, y_2, of these points or e. g. by the coordinates x_0, y_0, of the centre of the segment A_1A_2 and the angle φ which this segment makes with the x-axis. Knowing x_0, y_0, and φ, we determine the coordinates of the points A_1, A_2, from the formulae:

$$x_1 = x_0 - \tfrac{1}{2}d\cos\varphi, \quad y_1 = y_0 - \tfrac{1}{2}d\sin\varphi,$$
$$x_2 = x_0 + \tfrac{1}{2}d\cos\varphi, \quad y_2 = y_0 + \tfrac{1}{2}d\sin\varphi.$$

We can define the position of a rigid body in space similarly by choosing an arbitrary system of coordinates (ξ, η, ζ) attached rigidly to the body, and giving the coordinates x_0, y_0, z_0, of the origin of the system (ξ, η, ζ) with respect to a fixed system (x, y, z) as well as the angles $\alpha_1, \ldots, \gamma_3$, which the axes ξ, η, ζ, make with the axes x, y, z, of the fixed system. The coordinates of the points of the body are then determined by formulae (I), p. 53.

The position of a rigid body having a fixed axis is defined by one number φ denoting the angle through which it would be necessary to rotate the body about the axis in order that it pass from its initial position to the given position. Choosing the axis of rotation as the axis of z and denoting by ξ, η, ζ, the coordinates of an arbitrary point in the initial position, and by x, y, z, the coordinates of this point after a rotation through the angle φ, we have:

$$x = \xi \cos\varphi - \eta \sin\varphi, \quad y = \xi \sin\varphi + \eta \cos\varphi, \quad z = \zeta.$$

Let a holonomo-scleronomic system consisting of n material points be given.

Let us assume that to every position of a system of points which is compatible with the constraints and near the position investigated there corresponds a set of k numbers $q_1, \ldots, q_k$, in such a way that to different positions of the system there correspond different sets of numbers $q_1, \ldots, q_k$. From this assumption it follows that the coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, of the points of the system are functions of the variables $q_1, \ldots, q_k$:

$$x_1 = f_1(q_1, \ldots, q_k), \quad y_1 = \varphi_1(q_1, \ldots, q_k), \quad z_1 = \psi_1(q_1, \ldots, q_k),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = f_n(q_1, \ldots, q_k), \quad y_n = \varphi_n(q_1, \ldots, q_k), \quad z_n = \psi_n(q_1, \ldots, q_k).$$

We write the above equations more briefly in the form:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k), \quad (i = 1, 2, \ldots, n). \tag{I}$$

In general, we assume that the functions f_i, φ_i, ψ_i, are together with their partial derivatives continuous in a certain region of the variables $q_1, \ldots, q_k$, and moreover that to different sets of numbers $q_1, \ldots, q_k$, there correspond in this region different sets of numbers $x_1, \ldots, z_n$.

The numbers $q_1, \ldots, q_k$, are called the *parameters of the system* or *Lagrange's generalized coordinates.*

The coordinates $x_1, \ldots, z_n$, with respect to a certain inertial frame are called *natural coordinates* in order to distinguish them from Lagrange's generalized coordinates.

They are obviously a particular case of Lagrange's coordinates.

Parameters are said to be *independent* if to every set of the variables $q_1, \ldots, q_k$, the functions (I) correspond to positions of the system compatible with the constraints.

In the case of bilateral constraints we can in general choose independent parameters. For let the constraints of the system be defined by the functions:

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m). \tag{1}$$

From equations (1) we can in general determine m unknowns knowing the remaining $k = 3n - m$. Therefore choosing arbitrarily k variables from among $x_1, \ldots, z_n$, and denoting them by $q_1, \ldots, q_k$, we shall be able to represent the variables $x_1, \ldots, z_n$, as functions of the variables $q_1, \ldots, q_k$, in the form (I). Since to the arbitrarily chosen values $q_1, \ldots, q_k$, there correspond the variables $x_1, \ldots, z_n$, satisfying the system of equations (1), the parameters are independent.

The number $k = 3n - m$ was called the number of degrees of freedom of the system (p. 421).

Therefore: *the number of independent parameters is equal to the number of degrees of freedom of a system.*

If the parameters are dependent, then in the case of bilateral constraints certain relations

$$\Phi_j(q_1, \ldots, q_k) = 0 \qquad (j = 1, 2, \ldots, \varrho) \tag{2}$$

must hold among those parameters $q_1, \ldots, q_k$, which define the position of the system compatible with the constraints.

Choosing certain $k - \varrho$ parameters arbitrarily, with their aid we can determine the remaining parameters from relations (2). The parameters chosen will be independent parameters.

In the case of unilateral constraints the parameters defining the position of the system compatible with the constraints must satisfy, in addition to relations of the form (2), certain inequalities of the form

$$\Psi_r(q_1, \ldots, q_k) \leqq 0 \qquad (r = 1, 2, \ldots, s). \tag{3}$$

If the parameters are independent, then the functions (I) define the constraints of the system, because they give all of its positions compatible with the constraints. If the parameters are dependent, then in addition to functions (I) it is necessary to give relations (2) and (3), which the parameters corresponding to the positions of the system compatible with the constraints must satisfy.

Example 1. A point is constrained to lie on the surface of the sphere $x^2 + y^2 + z^2 = r^2$. Choosing arbitrary x and y, we have for the upper hemisphere $z = \sqrt{r^2 - x^2 - y^2}$. Hence if we put

$$x = q_1, \quad y = q_2, \quad \text{then} \quad z = \sqrt{r^2 - q_1^2 - q_2^2}. \tag{4}$$

The numbers q_1 and q_2 are independent parameters.

Example 2. A material point A is constrained to remain on the surface of the sphere $x^2 + y^2 + z^2 - r^2 = 0$. Denoting by q_1, q_2, q_3, the cosines of the angles made by the vector $\overline{OA}$ (where O denotes the origin of the coordinate system) with the coordinate axes, we have:

$$x = rq_1, \quad y = rq_2, \quad z = rq_3. \tag{5}$$

The parameters q_1, q_2, q_3, are dependent, for they must obviously satisfy the equation

$$q_1^2 + q_2^2 + q_3^2 - 1 = 0; \tag{6}$$

from this equation (when $z \geqq 0$) we obtain $q_3 = \sqrt{1 - q_1^2 - q_2^2}$, whence by substituting in (5):

$$x_1 = rq_1, \quad y = rq_2, \quad z = r\sqrt{1 - q_1^2 - q_2^2}. \tag{7}$$

The parameters q_1 and q_2 in representation (7) are independent.

Virtual displacements. Let the constraints be defined parametrically by the functions:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k), \tag{II}$$
$$(i = 1, 2, \ldots, n).$$

Let us assume that the parameters are independent.

If we give the system an arbitrary motion compatible with the constraints, then $q_1, \ldots, q_k$ will be functions of the time. Differentiating (II), we obtain:

$$\dot{x_i} = \frac{\partial f_i}{\partial q_1}\dot{q_1} + \ldots + \frac{\partial f_i}{\partial q_k}\dot{q_k}, \qquad \dot{y_i} = \frac{\partial \varphi_i}{\partial q_1}\dot{q_1} + \ldots + \frac{\partial \varphi_i}{\partial q_k}\dot{q_k},$$
$$\dot{z_i} = \frac{\partial \psi_i}{\partial q_1}\dot{q_1} + \ldots + \frac{\partial \psi_i}{\partial q_k}\dot{q_k}, \qquad (i = 1, 2, \ldots, n). \tag{8}$$

Conversely, if we assume that $q_1, \ldots, q_k$, are arbitrary functions of time, the formulae (II) obviously define the motion of the system compatible with the constraints; hence equations (8) give the velocities of the points of the system in this motion. It follows from this that if the system is in a certain position defined by the parameters $q_1, \ldots, q_k$, then all the systems of possible velocities in this position are obtained from (8) by substituting arbitrary values for $\dot{q_1}, \ldots, \dot{q_k}$. Assuming:

$$\delta x_1 = \dot{x_1}, \quad \ldots, \quad \delta z_n = \dot{z_n},$$
$$\delta q_1 = \dot{q_1}, \quad \ldots, \quad \delta q_k = \dot{q_k},$$

and writing $\frac{\partial x_i}{\partial q_1}$ instead of $\frac{\partial f_i}{\partial q_1}$, etc., we obtain from (8):

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\delta q_k, \qquad \delta y_i = \frac{\partial y_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial y_i}{\partial q_k}\delta q_k,$$
$$\delta z_i = \frac{\partial z_i}{\partial q_1}\delta q_1 + \ldots + \frac{\delta z_i}{\partial q_k}\delta q_k \qquad (i = 1, 2, \ldots n). \tag{III}$$

Substituting arbitrary values for $\delta q_1, \ldots, \delta q_k$, in formulae (III), we obtain the virtual displacements of a system. Conversely, every virtual displacement of a system is obtained by the substitution of suitable values of $\delta q_1, \ldots, \delta q_k$.

Let us note that the system of formulae (III) can be obtained by forming the derivatives of the system of formulae (II) formally, and then writing $\delta x_i, \delta y_i, \delta z_i$, instead of $\mathrm{d}x_i, \mathrm{d}y_i, \mathrm{d}z_i$, and $\delta q_1, \ldots, \delta q_k$, instead of $\mathrm{d}q_1, \ldots, \mathrm{d}q_k$.

In example 1, p. 453, we have from formulae (4):

$$\delta x = \delta q_1, \quad \delta y = \delta q_2, \quad \mathrm{d}z = -(q_1\, \delta q_1 + q_2\, \delta q_2) \,/\, \sqrt{1 - q_1^2 - q_2^2}.$$

Choosing the values of δq_1 and δq_2 arbitrarily, we obtain the virtual displacements $\delta x, \delta y, \delta z$, in the position corresponding to the parameters q_1 and q_2.

If $q_1, \ldots, q_k$, are dependent parameters and relations of the form (2), p. 453, hold among them, then $\delta q_1, \ldots, \delta q_k$, are not arbitrary numbers in formulae (III), but — as can be shown (cf. the proof of formula (15), p. 426) — they must satisfy the system of equations

$$\frac{\partial \Phi_j}{\partial q_1}\,\delta q_1 + \ldots + \frac{\partial \Phi_j}{\partial q_k}\,\delta q_k = 0 \qquad (j = 1, 2, \ldots, \varrho). \qquad \text{(IV)}$$

In example 2, p. 453, we have from formulae (5):

$$\delta x = r\, \delta q_1, \quad \delta y = r\, \delta q_2, \quad \delta z = r\, \delta q_3,$$

where by (6), p. 454, the relation

$$q_1\, \delta q_1 + q_2\, \delta q_2 + q_3\, \delta q_3 = 0 \text{ holds among } \delta q_1, \delta q_2, \delta q_3.$$

It can be shown (cf. the proof of formula (II), p. 432) that if the constraints are unilateral and in addition to relations (2), p. 453, relations (3) hold, (p. 453), then $\delta q_1, \ldots, \delta q_k$, must satisfy besides (IV) those relations from among

$$\frac{\partial \Psi_r}{\partial q_1}\,\delta q_1 + \ldots + \frac{\partial \Psi_r}{\partial q_k}\,\delta q_k \leqq 0 \qquad (r = 1, 2, \ldots, s), \qquad \text{(V)}$$

for which the equality $\Phi_r = 0$ holds in a given position of the system.

Virtual work. Generalized forces. Let the constraints of a system be defined parametrically by the equations:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k), \qquad (9)$$
$$(i = 1, 2, \ldots, n).$$

The virtual displacements are expressed by formulae (III), p. 454.

If the parameters are independent, then $\delta q_1, \ldots, \delta q_k$, are arbitrary numbers. In the contrary case certain relations (IV) and (V) hold among them.

Substituting in (I′), p. 434, the expressions (III), p. 454, for δx_i, δy_i, δz_i we obtain

$$\delta' L = \sum_{i=1}^{n} \left[P_{i_x} \left(\frac{\partial x_i}{\partial q_1} \delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k} \delta q_k \right) + \right.$$
$$\left. + P_{i_y} \left(\frac{\partial y_i}{\partial q_1} \delta q_1 + \ldots + \frac{\partial y_i}{\partial q_k} \delta q_k \right) + P_{i_z} \left(\frac{\partial z_i}{\partial q_1} \delta q_1 + \ldots + \frac{\partial z_i}{\partial q_k} \delta q_k \right) \right]. \tag{10}$$

Arranging the terms according to $\delta q_1, \ldots, \delta q_k$, we get

$$\delta q_1 \sum_{i=1}^{n} \left(P_{i_x} \frac{\partial x_i}{\partial q_1} + P_{i_y} \frac{\partial y_i}{\partial q_1} + P_{i_z} \frac{\partial z_i}{\partial q_1} \right) + \ldots +$$
$$+ \delta q_k \sum_{i=1}^{n} \left(P_{i_x} \frac{\partial x_i}{\partial q_n} + P_{i_y} \frac{\partial y_i}{\partial q_k} + P_{i_z} \frac{\partial z_i}{\partial q_k} \right).$$

Denoting by $Q_1, \ldots, Q_k$, the sums appearing as coefficients of $\delta q_1, \ldots, \delta q_k$:

$$Q_1 = \sum_{i=1}^{n} \left(P_{i_x} \frac{\partial x_i}{\partial q_1} + P_{i_y} \frac{\partial y_i}{\partial q_1} + P_{i_z} \frac{\partial z_i}{\partial q_1} \right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{VI}$$
$$Q_k = \sum_{i=1}^{n} \left(P_{i_x} \frac{\partial x_i}{\partial q_k} + P_{i_y} \frac{\partial y_i}{\partial q_k} + P_{i_z} \frac{\partial z_i}{\partial q_k} \right),$$

which we write more briefly as

$$Q_j = \sum_{i=1}^{n} \left(P_{i_x} \frac{\partial x_i}{\partial q_j} + P_{i_y} \frac{\partial y_i}{\partial q_j} + P_{i_z} \frac{\partial z_i}{\partial q_j} \right) \qquad (j = 1, 2, \ldots, k). \tag{VI′}$$

We obtain from this in virtue of (10) $\delta' L = Q_1 \, \delta q_1 + \ldots + Q_k \, \delta q_k$, i.e.

$$\delta' L = \sum_{j=1}^{k} Q_j \, \delta q_j. \tag{VII}$$

The expressions $Q_1, \ldots, Q_k$, defined by formulae (VI) and (VI′) are called the *components of the generalized force* or briefly *generalized forces*.

Comparing formula (VII) with formula (I′), p. 434, we see that the virtual work is expressed by means of the components Q_j of the generalized forces and by means of the displacements δq_j in a similar manner as by means of the components $P_{i_x}, P_{i_y}, P_{i_z}$, and the displacements $\delta x_i, \delta y_i, \delta z_i$.

Conditions of equilibrium. Let the constraints of a holonomo-scleronomic system be defined be means of the parameters $q_1, \ldots, q_k$. By (VII) condition (II), p. 437, of the equilibrium of a system assumes the form

$$\delta' L = \sum_{j=1}^{k} Q_j \, \delta q_j \leqq 0. \tag{VIII}$$

If the constraints are bilateral, then ((III), p. 437)

$$\delta' L = \sum_{j=1}^{k} Q_j\,\delta q_j = 0. \tag{IX}$$

The generalized forces $Q_1, \ldots, Q_k$, appearing in formulae (VIII) and (IX) are defined by formulae (VI) and (VI').

If the parameters $q_1, \ldots, q_k$, are independent, then $\delta q_1, \ldots, \delta q_k$, are arbitrary. Hence by (IX), taking $\delta q_1 = 1$ and $\delta q_2 = \delta q_3 = \ldots = \delta q_k = 0$, we get $Q_1 = 0$ and similarly $Q_2 = 0, \ldots, Q_k = 0$.

Therefore: *if the parameters defining the position of a holonomo-scleronomic system are independent (and there is no friction), then the necessary and sufficient condition for the equilibrium of the acting forces is that the generalized forces be equal to zero*:

$$Q_j = 0 \qquad (j = 1, 2, \ldots, k). \tag{X}$$

If the parameters $q_1, \ldots, q_k$, are dependent and satisfy relations (2) or (3), p. 453, then $\delta q_1, \ldots, \delta q_k$, are not arbitrary numbers: they must satisfy relations (IV) or (V), p. 455. The position of equilibrium is then determined in the same way as for the natural coordinates, i. e. in the manner given on p. 445.

Remark. Natural coordinates are a particular case of generalized coordinates. Therefore, if the natural coordinates of the points of a system are $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, then putting:

$$x_1 = q_1, \quad y_1 = q_2, \quad z_1 = q_3, \quad \ldots, \quad z_n = q_{3n},$$

we can apply the formulae (I)—(X) of the present §. The results obtained in this § are hence a generalization of the corresponding results for the natural coordinates.

Example 3. Four rods of equal length a, lying in a vertical plane, are pin-connected at A, B, C, and D. The joint A is fixed and C can move only along a vertical line l passing through A. Horizontal forces $\mathbf{P}$ and $-\mathbf{P}$ act at the joints B and D, while a vertical force $\mathbf{Q}$ (having a downward sense) acts at C. Determine the position of equilibrium neglecting the weights of the rods and assuming that there is no friction and that the rods can move only in the vertical plane in which the forces $\mathbf{P}$ and $\mathbf{Q}$ lie (Fig. 313).

Let us take A as the origin of the coordinate system (x, y) of the vertical plane in which the rods lie and the line l as the y-axis (with a downward sense). Then the position of the system will be defined by giving the angle α between the rod AB and the y-axis. Consequently α is a parameter of the system.

From Fig. 313 it is seen that the coordinates x_1, y_1, x_2, y_2, and x_3, y_3, of the joints B, C, and D, are the following:

$$\begin{aligned} x_1 &= -a\sin\alpha, \quad x_2 = 0, \qquad\quad x_3 = a\sin\alpha, \\ y_1 &= a\cos\alpha, \qquad y_2 = 2a\cos\alpha, \; y_3 = a\cos\alpha. \end{aligned} \tag{11}$$

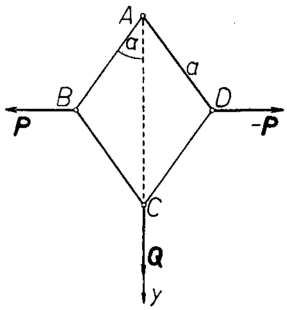

Fig. 313.

The virtual work is

$$\delta' L = -P\,\delta x_1 + Q\,\delta y_2 + P\,\delta x_3, \tag{12}$$

where $P = |\mathbf{P}|$ and $Q = |\mathbf{Q}|$. By (11) we have:

$$\begin{aligned} \delta x_1 &= -a\cos\alpha\,\delta\alpha, \\ \delta y_2 &= -2a\sin\alpha\,\delta\alpha, \\ \delta x_3 &= a\cos\alpha\,\delta\alpha. \end{aligned}$$

Substituting these values in (12), we get

$$\delta' L = 2a(P\cos\alpha - Q\sin\alpha)\,\delta\alpha. \tag{13}$$

In the position of equilibrium $\delta' L = 0$, hence $2a(P\cos\alpha - Q\sin\alpha)\,\delta\alpha = 0$; consequently $P\cos\alpha - Q\sin\alpha = 0$; whence

$$\tan\alpha = P / Q.$$

Example 4. A system of rods $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$, pin-connected at $A_1, A_2, \ldots, A_{n-1}$, is given. Forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, with origins at $A_1, A_2, \ldots, A_n$, act on the system. The point A_0 is fixed. Determine the position of equilibrium, assuming that the rods and the forces lie in one plane.

Let us take the point A_0 as the origin of the coordinate system (x, y) of this plane and denote by $a_1, a_2, \ldots, a_n$, the lengths of the rods, finally by $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, the coordinates of the points $A_1, A_2, \ldots, A_n$ (Fig. 314). We have:

$$\begin{gathered} x_1 = a_1\cos\alpha_1, \; x_2 = a_1\cos\alpha_1 + a_2\cos\alpha_2, \; \ldots, \\ x_n = a_1\cos\alpha_1 + \ldots + a_n\cos\alpha_n, \end{gathered} \tag{14}$$

$$\begin{gathered} y_1 = a_1\sin\alpha_1, \; y_2 = a_1\sin\alpha_1 + a_2\sin\alpha_2, \; \ldots, \\ y_n = a_1\sin\alpha_1 + \ldots + a_n\sin\alpha_n. \end{gathered} \tag{15}$$

From (14) and (15) it follows that the angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, define the position of the system of rods. Consequently the variables $\alpha_1, \ldots, \alpha_n$, are the parameters of the system, and since they are not related in any way, they are independent parameters.

Differentiating (14) and (15), we obtain:

$$\delta x_1 = -a_1 \sin\alpha_1\, \delta\alpha, \ldots, \delta x_n = -a_1 \sin\alpha_1\, \delta\alpha_1 - \ldots - a_n \sin\alpha_n\, \delta\alpha_n, \quad (16)$$

$$\delta y_1 = \ \ a_1 \cos\alpha_1\, \delta\alpha_1, \ldots, \delta y_n = \ \ a_1 \cos\alpha_1\, \delta\alpha_1 + \ldots + a_n \cos\alpha_n\, \delta\alpha_n. \quad (17)$$

In the position of equilibrium the virtual work is

$$\delta' L = (P_{1_x}\, \delta x_1 + P_{1_y}\, \delta y_1) + \ldots + (P_{n_x}\, \delta x_n + P_{n_y}\, \delta y_n) = 0. \quad (18)$$

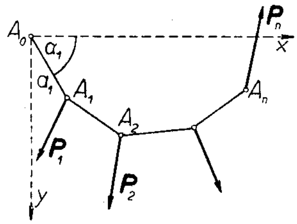

Fig. 314.

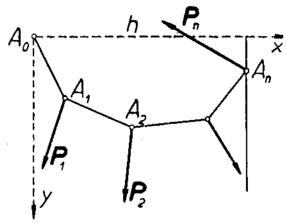

Fig. 315.

Substituting the values (16) and (17) in (18), we get after arranging terms

$$\begin{aligned} &a_1[-(P_{1_x} + \ldots + P_{n_x}) \sin\alpha_1 + (P_{1_y} + \ldots + P_{n_y}) \cos\alpha_1]\, \delta\alpha_1 + \\ &+ a_2[-(P_{2_x} + \ldots + P_{n_x}) \sin\alpha_2 + (P_{2_y} + \ldots + P_{n_y}) \cos\alpha_2]\, \delta\alpha_2 + \quad (19) \\ &+ \ldots + a_n[-P_{n_x} \sin\alpha_n + P_{n_y} \cos\alpha_n]\, \delta\alpha_n = 0. \end{aligned}$$

The coefficients of $\delta\alpha_1, \ldots, \delta\alpha_n$, are the generalized forces $Q_1, \ldots, Q_n$. Since equality (19) holds for every set of numbers $\delta\alpha_1, \ldots, \delta\alpha_n$, the generalized forces are zero. Consequently:

$$\begin{aligned} &-(P_{1_x} + \ldots + P_{n_x}) \sin\alpha_1 + (P_{1_y} + \ldots + P_{n_y}) \cos\alpha_1 = 0, \\ &-(P_{2_x} + \ldots + P_{n_x}) \sin\alpha_2 + (P_{2y} + \ldots + P_{n_y}) \cos\alpha_2 = 0, \quad (20) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &-P_{n_x} \sin\alpha_n + P_{n_y} \cos\alpha_n = 0. \end{aligned}$$

From equations (20) it is easy to calculate the tangents of the angles $\alpha_1, \ldots, \alpha_n$. Since the tangents are equal for angles differing by 180°, we obtain many solutions.

If the coefficients of the sines and cosines in one of the equations (20) are zero, then the corresponding angle can be chosen arbitrarily.

Let us now assume that the point A_n is to remain on a line l having the equation $x = h$, which is the case e. g. when the end A_n of the rod $A_{n-1}A_n$ is a ring that slides on a rigid wire having the position of the line l (Fig. 315). Under this assumption the parameters $\alpha_1, \ldots, \alpha_n$, will not be independent, since the relation $x_n = h$ will have to hold, i. e. in view of (14)

$$a_1 \cos \alpha_1 + \ldots + a_n \cos \alpha_n - h = 0. \tag{21}$$

Differentiating (21), we obtain

$$-a_1 \sin \alpha_1 \, \delta \alpha_1 + \ldots - a_n \sin \alpha_n \, \delta \alpha_n = 0; \tag{22}$$

by substituting arbitrary values for $\delta\alpha_1, \ldots, \delta\alpha_{n-1}$, in (22) we obtain

$$\delta\alpha_n = -(a_1 \sin \alpha_1 \, \delta\alpha_1 + \ldots + a_{n-1} \sin \alpha_{n-1} \, \delta\alpha_{n-1}) \,/\, a_n \sin \alpha_n. \tag{23}$$

In view of (22) equation (19) assumes the form

$$\begin{aligned} &a_1[-(P_{1_n} + \ldots + P_{n-1,x}) \sin\alpha_1 + (P_{1_y} + \ldots + P_{n_y}) \cos\alpha_1] \, \delta\alpha_1 + \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_{n-1}[-P_{n-1,x} \sin \alpha_{n-1} + (P_{n-1,y} + P_{n_y}) \cos \alpha_{n-1}] \, \delta\alpha_{n-1} + \\ &+ a_n P_{n_y} \cos \alpha_n \, \delta\alpha_n = 0. \end{aligned} \tag{24}$$

Substituting the value from (23) in (24), we get

$$\begin{aligned} &-a_1(P_{1_x} + \ldots + P_{n-1,x} + P_{n_y} \cot \alpha_n) \sin \alpha_1 \, \delta\alpha_1 + \\ &+ a_1 (P_{1_y} + \ldots + P_{n_y}) \cos\alpha_1 \, \delta\alpha_1 + \ldots + \\ &+ a_{n-1}[-(P_{n-1,x} + P_{n_y} \cot \alpha_n) \sin \alpha_{n-1} + (P_{n-1,y} + P_{n_y}) \cos \alpha_{n-1}] \, \delta\alpha_{n-1} = 0. \end{aligned}$$

Since $\delta\alpha_1, \ldots, \delta\alpha_{n-1}$ are arbitrary numbers, their coefficients are zero. We therefore obtain a system of $n - 1$ equations:

$$\begin{aligned} &-(P_{1_x} + \ldots + P_{n-1,x} + P_{n_y} \cot \alpha_n) \sin \alpha_1 + (P_{1_y} + \ldots + P_{n_y}) \cos \alpha_1 = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &-(P_{n-1,x} + P_{n_y} \cot \alpha_n) \sin \alpha_{n-1} + (P_{n-1,y} + P_{n_y}) \cos \alpha_{n-1} = 0. \end{aligned}$$

These equations together with equations (21) constitute a system of n equations from which we can determine n unknowns $\alpha_1, \ldots, \alpha_n$.

Example 5. A heavy ring K of mass m_1 slides on a curve C lying in the vertical plane xy. A string (massless and inextensible) passes through the ring; one of the ends of the string is tied at the origin O of the coordinate system; a heavy point A of mass m_2 is carried at the other end (Fig. 316). Determine the position of equilibrium, assuming that there is no friction.

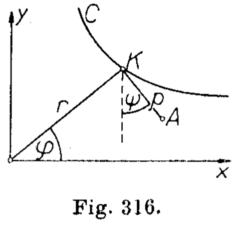

Fig. 316.

Let the equation of the curve C in polar coordinates be

$$r = f(\varphi). \tag{25}$$

The coordinates x_1, y_1, of the point K will therefore be:

$$x_1 = r\cos\varphi,\ y_1 = r\sin\varphi. \tag{26}$$

Let us put $\varrho = AK$. Denoting the length of the string by l we have

$$r + \varrho - l \leqq 0. \tag{27}$$

Let ψ be the angle between AK and the vertical, and x_2, y_2, the coordinates of the point A. Consequently $x_2 = x_1 + \varrho\sin\psi$, $y_2 = y_1 - \varrho\cos\psi$, whence by (26):

$$x_2 = r\cos\varphi + \varrho\sin\psi, \quad y_2 = r\sin\varphi - \varrho\cos\psi. \tag{28}$$

Equations (26) and (28) define the constraints of the system in terms of the parameters $r, \varrho, \varphi, \psi$, among which the relations (25) and (27) hold.

The virtual work is $\delta' L = -m_1 g\,\delta y_1 - m_2 g\,\delta y_2$. In the position of equilibrium $\delta' L \leqq 0$; hence

$$m_1\,\delta y_1 + m_2\,\delta y_2 \geqq 0. \tag{29}$$

When the string is not in tension the point A is free; hence δy_2 can be arbitrary. Taking $\delta y_1 = 0$ and $\delta y_2 < 0$ in this case, we should obtain $m_1\,\delta y_1 + m_2\,\delta y_2 < 0$ contrary to (29), which proves that the system cannot be in equilibrium when the string is not in tension.

Let us assume, therefore, that the string is in tension (i. e. that the equality sign holds in (27)) as well as that $r > 0$ and $\varrho > 0$; from (26) and (28) we obtain:

$$\begin{gathered} \delta y_1 = \delta r\sin\varphi + r\cos\varphi\,\delta\varphi, \\ \delta y_2 = \delta r\sin\varphi + r\cos\varphi\,\delta\varphi - \delta\varrho\cos\psi + \varrho\,\delta\psi\sin\psi. \end{gathered} \tag{30}$$

By (25) and (27) the following relations hold among δr, $\delta\varphi$, and $\delta\varrho$:

$$\delta r = f'(\varphi)\,\delta\varphi, \quad \delta r + \delta\varrho \leqq 0, \tag{31}$$

while $\delta\psi$ is arbitrary.

Substituting for δy_1, δy_2, in (29) the expressions from (30), we obtain

$$\begin{gathered} (m_1 + m_2)\sin\varphi\,\delta r + (m_1 + m_2)\,r\cos\varphi\,\delta\varphi - m_2\,\delta\varrho\cos\psi + \\ + m_2\varrho\,\delta\psi\sin\psi \geqq 0. \end{gathered} \tag{32}$$

In virtue of (31) we can assume that $\delta r = 0$, $\delta\varphi = 0$, and $\delta\varrho = 0$, whence by (32)

$$m_2\varrho\,\delta\psi\sin\psi \geqq 0. \tag{33}$$

Since $\delta\psi$ is arbitrary, inequality (33) will hold only when $m_2\varrho\sin\psi = 0$ or when $\sin\psi = 0$, and therefore for $\psi = 0$ and $\psi = \pi$. It is intuitively evident that in the position of equilibrium we can only have $\psi = 0$, for the equality $\psi = \pi$ means that A is above K, which is obviously impossible in the position of equilibrium. This also follows from relation (32), for by (31) assuming $\delta r = 0$, $\delta\varphi = 0$, $\delta\varrho < 0$, and $\delta\psi = 0$, we should have from (32) $-m_2\,\delta\varrho\cos\psi \geqq 0$, and since $\delta\varrho < 0$, $\cos\psi \geqq 0$, whence $\psi \neq \pi$.

Consequently we have proved that $\psi = 0$ in the position of equilibrium, i. e. that A is below K. Substituting $\psi = 0$ in (32), we obtain by (25) and (31)

$$(m_1 + m_2)(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi)\,\delta\varphi - m_2\,\delta\varrho \geqq 0. \tag{34}$$

By (31) we can put ϱ

$$\delta r + \delta\varrho = 0, \text{ whence } \delta\varrho = -\delta r = -f'(\varphi)\,\delta\varphi,$$

where $\delta\varphi$ is arbitrary. Substituting in (34), we get then

$$[(m_1 + m_2)(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi) + m_2 f'(\varphi)]\,\delta\varphi \geqq 0.$$

Since $\delta\varphi$ is arbitrary, this relation holds only when

$$(m_1 + m_2)(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi) + m_2 f'(\varphi) = 0. \tag{35}$$

Equation (35) enables one to determine the angle φ in the position of equilibrium.

Conversely, if equation (35) holds, then inequality (34) must be satisfied. For let us denote by W the left side of this inequality. By (35) $W = -m_2 f'(\varphi)\,\delta\varphi - m_2\,\delta\varrho$, whence by (31) $W = -m_2(\delta r + \delta\varrho)$. Since, because of (31), $\delta r + \delta\varrho \leqq 0$, whence $W \geqq 0$. Therefore, if the angle φ satisfies equation (35), then equilibrium occurs.

We shall yet investigate for what curve C equilibrium occurs for every value of φ, i. e. the case when equation (35) becomes an identity.

Let us note in this connection that the left side of equation (35) is the derivative of the function $f(\varphi)[(m_1 + m_2)\sin\varphi + m_2]$; consequently

$$f(\varphi)[(m_1 + m_2)\sin\varphi + m_2] = c = \text{const.}$$

Since $r = f(\varphi)$ by (1),

$$r = \frac{c}{(m_1 + m_2)\sin\varphi + m_2}. \tag{36}$$

If $c \neq 0$, $m_1 > 0$ and $m_2 > 0$, then the above equation is the equation of the lower branch of the hyperbola whose real axis is the y-axis.

Equilibrium in a potential field. Let us assume that the forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, have the potential V. Consequently ((III), p. 211):

$$P_{i_x} = \frac{\partial V}{\partial x_i}, \quad P_{i_y} = \frac{\partial V}{\partial y_i}, \quad P_{i_z} = \frac{\partial V}{\partial z_i}. \tag{37}$$

Substituting these values in formula (VI′), p. 456, we get

$$Q_j = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} \right). \tag{38}$$

The potential V is a function of the variables $x_1, \ldots, z_n$. Expressing them in terms of the parameters $q_1, \ldots, q_k$, we can therefore assume that V is a function of the parameters $q_1, \ldots, q_k$. From a well-known formula in differential calculus on the partial derivative of a compound function, it follows that the right side of equation (38) is the partial derivative $\partial V / \partial q_j$. Consequently

$$Q_j = \partial V / \partial q_j \qquad (j = 1, 2, \ldots, k). \tag{XI}$$

Comparing (37) and (XI) we see that the components of the generalized forces are expressed similarly as the components of forces relative to the natural coordinates.

Therefore: *if a force field is a potential field, then the components of the generalized forces are the partial derivatives of the potential with respect to the generalized coordinates.*

From formulae (VII), p. 456, and (XI) we obtain

$$\delta' L = \sum_{j=1}^{k} \frac{\partial V}{\partial q_j} \delta q_j. \tag{39}$$

By (18), p. 428, we can therefore write

$$\delta' L = \delta V, \tag{XII}$$

where V is considered as a function of the variables $q_1, \ldots, q_k$.

Formula (XII) has the same form as formula (I″), p. 434. The difference consists in the fact that in formula (I″) we consider V as a function of the natural coordinates $x_1, \ldots, z_n$, while in formula (XII) we consider V as a function of the parameters $q_1, \ldots, q_k$.

Remark. The meaning of the expression δV is illustrated as follows. At the moment t let us give the system in a given position an arbitrary motion compatible with the constraints. The parameters $q_1, \ldots, q_k$, as well as the potential V, will therefore be functions of the time. We have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{j=1}^{k} \frac{\partial V}{\partial q_j} q^{\cdot}_j. \tag{40}$$

Since we can assume that $q^{\cdot}_j = \delta q_j$ for $j = 1, 2, \ldots$, by (40) we have $\delta V = \mathrm{d}V / \mathrm{d}t$. In this way δV represents the rate of change (i. e. the derivative) of the potential for an arbitrarily given motion of the system compatible with the constraints.

From the principle of virtual work it follows by (XII) that the necessary and sufficient condition for the equilibrium of a system is that

$$\delta V \leqq 0. \tag{XIII}$$

In particular, therefore, *a position of equilibrium of a system is a position in which the potential attains a maximum, and in the case of bilateral constraints a position in which the potential attains a minimum.*

For if we give a system an arbitrary motion compatible with the constraints at the moment when V attains a maximum, then after a time Δt the increase in the potential will be $\Delta V \leqq 0$, from which $\mathrm{d}V / \mathrm{d}t \leqq 0$ (where the inequality $\mathrm{d}V / \mathrm{d}t < 0$ may hold in the boundary position for unilateral constraints), and consequently $\delta V \leqq 0$ (in view of the remark on the meaning of δV). If the constraints are bilateral and V has a minimum value, then $\mathrm{d}V / \mathrm{d}t = 0$, whence $\delta V = 0$.

If the only forces acting on a system of material points are the gravitational forces, then the potential is ((13), p. 211)

$$V = -mgz_0, \tag{41}$$

where m denotes the total mass of the system, and z_0 the coordinate of the centre of mass, under the assumption that the axis of z has a sense vertically upwards.

The position of equilibrium will therefore be every position at which V is a maximum or z_0 a minimum. If the constraints are bilateral, then the position of equilibrium will be in addition to this every position in which V is a minimum or z_0 a maximum.

If a heavy point A hangs on an inextensible string tied at the point O, then the extrema of the potential V occur when the string is in tension and has a vertical direction. A maximum occurs when A is below O, and a minimum occurs when A is above O. It is obvious that the position of equilibrium occurs only when V has a maximum value (i. e. when A is under O).

Example 6. Two heavy material points A_1 and A_2 of masses m_1 and m_2 are connected by an inextensible string passing over a pulley. The point m_2 is constrained to remain on a vertical line l. What angle does the string make with the line l in the position of equilibrium, if there is no friction (Fig. 130)?

Let us take the line l as the axis of z,, giving it an upward sense, and the point of the axis which is at the top of the pulley as the origin of the coordinate system. Let us denote by s the length of the string, by a the distance of the pulley from the axis l, by z_1 and z_2 the coordinates of the points A_1 and A_2, and by z_0 the coordinate of the centre of mass. We have:

$$z_1 = -(s - a / \sin\varphi), \quad z_2 = -a \cot\varphi.$$

Since $mz_0 = m_1 z_1 + m_2 z_2$, where $m = m_1 + m_2$,

$$z_0 = [-m_0(s - a / \sin\varphi) - m_2 a \cot\varphi] / m.$$

In order to determine an extremum of z_0, let us set the derivative $dz_0 / d\varphi$ equal to zero:

$$-m_1 a \cos\varphi / \sin^2\varphi + m_2 a / \sin^2\varphi = 0,$$

whence

$$\cos\varphi = m_2 / m_1. \tag{42}$$

It is easy to show that z_0 is a minimum for the value of φ satisfying equation (42). Therefore equation (42) defines the position of equilibrium (when $m_2 < m_1$).

Another way of solving this problem is given in example 2, p. 191.