CHAPTER XI

VARIATIONAL PRINCIPLES OF MECHANICS

§ I. Variation without the variation of time. In this paragraph we shall give certain information from the calculus of variations necessary for the understanding of what follows.

Variation of a function. Let us take under consideration the motion of a point along the x axis, defined by the function

$$x = x(t) \qquad (t_0 \leqq t \leqq t_1). \quad (1)$$

Let there be given a function

$$T = F(x, x^{\cdot}, t) \quad (2)$$

continuous and having continuous partial derivatives of the first and second orders in a certain region D of the variables x, $x^{\cdot}$, t.

Let us next take under consideration an arbitrary motion along the x-axis, defined by the function

$$\mathbf{x} = \mathbf{x}(t) \qquad (t_0 \leqq t \leqq t_1). \quad (3)$$

Let us assume that it is possible to choose a number $\varepsilon > 0$ such that if

$$|\mathbf{x}(t) - x(t)| < \varepsilon, \quad |\mathbf{x}^{\cdot}(t) - x^{\cdot}(t)| < \varepsilon \qquad (t_0 \leqq t \leqq t_1), \quad (4)$$

then the function T and its partial derivatives will be continuous functions in the interval $\langle t_0, t_1 \rangle$ when $\mathbf{x}(t)$ and $\mathbf{x}^{\cdot}(t)$ are substituted in (2) for x and $x^{\cdot}$, respectively.

Let us put:

$$\delta x = \mathbf{x} - x, \quad \delta x^{\cdot} = \mathbf{x}^{\cdot} - x^{\cdot}, \quad (5)$$

where x and $\mathbf{x}$ denote the functions $x(t)$ and $\mathbf{x}(t)$. Consequently δx and $\delta x^{\cdot}$ are functions of the time t, defined in the interval $\langle t_0, t_1 \rangle$.

One should note the difference in the meaning of the symbol δx in chapters IX, X and now. Before, the symbol δx denoted a number, and now it denotes a function of the time t.

By (5) we have:

$$\delta x^{\cdot} = \frac{\mathrm{d}(\delta x)}{\mathrm{d}t}, \tag{I}$$

$$\mathbf{x} = x + \delta x, \quad \mathbf{x}^{\cdot} = x^{\cdot} + \delta x^{\cdot}. \tag{6}$$

Let

$$\mathbf{T} = F(\mathbf{x}, \mathbf{x}^{\cdot}, t) = F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t),$$

where x denotes the function (1), and δx is defined by (5). From Taylor's formula we get

$$\begin{aligned} \mathbf{T} - T &= F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t) - F(x, x^{\cdot}, t) = \\ &= \frac{\partial T}{\partial x}\,\delta x + \frac{\partial T}{\partial x^{\cdot}}\,\delta x^{\cdot} + R. \end{aligned} \tag{7}$$

The remainder R can be written in the form

$$R = (|\delta x| + |\delta x^{\cdot}|)\,\eta, \tag{8}$$

where η is a function of the time t and depends on x, δx, $\delta x^{\cdot}$, and where η tends to zero uniformly when the functions δx and $\delta x^{\cdot}$ tend to zero uniformly. Therefore, if $|\delta x|$ and $|\delta x^{\cdot}|$ are small, then $|\eta|$ is small, and consequently $|R|$ is of a still higher order of smallness. We express this briefly by saying that R is *"infinitesimal"* as compared with $|\delta x| + |\delta x^{\cdot}|$.

Let us put

$$\delta T = \frac{\partial T}{\partial x}\,\delta x + \frac{\partial T}{\partial x^{\cdot}}\,\delta x^{\cdot}. \tag{II}$$

Hence by (7)

$$\mathbf{T} - T = \delta T + R. \tag{9}$$

The expression δT is called the *variation of the function* $T = F(x, x^{\cdot}, t)$ *at the place* $x = x(t)$ or *for the function* $x = x(t)$.

In formula (II) the function δx is an arbitrary function of time having continuous first and second derivatives in the interval $\langle t_0, t_1 \rangle$. This follows from (5), where $\mathbf{x}$ is an arbitrary function having continuous first and second derivatives. In virtue of (I) the symbol $\delta x^{\cdot}$ denotes the derivative of the function δx with respect to the time t.

The variation δT therefore depends on the functions x and δx.

For purposes of differentiation we shall call δx *the variation of the independent variable* (*or of the function*) x, and δT *the variation of the dependent variable* (*or of the function*) T.

The motion $\mathbf{x} = \mathbf{x}(t) = x + \delta x$, will be called a *comparative motion*.

The variation δT therefore denotes approximately the increment of the function T when we pass from a point in the given motion at the moment t

to a point in the comparative motion at the same moment t. In virtue of (8) the difference R between the variation δT and the true increment $\mathbf{T} - T$ is "infinitesimal" as compared with the sum $|\delta x| + |\delta x^{\cdot}|$.

Since we are investigating the increment of the function T in the given motion and in the comparative motion at the same instant t, the variation δT is also called the *variation without the variation of time* in order to differentiate it from another kind of variation with which we shall meet later.

The variation δT is obtained by forming formally the differential of the function $F(x, x^{\cdot}, t)$ under the assumption that $t = \text{const}$ (i. e. $\mathrm{d}t = 0$) and then writing δ instead of d. We often write $\delta F(x, x^{\cdot}, t)$ instead of δT or briefly δF.

Example I. Let

$$T = \alpha x^2 + \beta x^{\cdot 2} t + \gamma t^2,$$

where α, β, γ, are constants. We have:

$$\frac{\partial T}{\partial x} = 2\alpha x, \qquad \frac{\partial T}{\partial x^{\cdot}} = 2\beta x^{\cdot} t;$$

consequently by (II)

$$\delta T = 2\alpha x \, \delta x + 2\beta x^{\cdot} t \, \delta x^{\cdot},$$

where δx is an arbitrary function.

Variation of an integral. Let us consider the integral

$$I = \int_{t_0}^{t_1} F(x, x^{\cdot}, t) \, \mathrm{d}t \tag{10}$$

and let

$$\mathbf{I} = \int_{t_0}^{t_1} F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t) \, \mathrm{d}t.$$

We have

$$\mathbf{I} - I = \int_{t_0}^{t_1} [F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t) - F(x, x^{\cdot}, t)] \, \mathrm{d}t,$$

and hence by (7) and (II)

$$\mathbf{I} - I = \int_{t_0}^{t_1} \delta F \, \mathrm{d}t + \int_{t_0}^{t_1} R \, \mathrm{d}t. \tag{11}$$

The expression

$$\int_{t_0}^{t_1} \delta F \, \mathrm{d}t$$

is called the *variation of the integral* (10) and we denote it by δI.

Therefore according to the definition

$$\delta I = \delta \int_{t_0}^{t_1} F \, \mathrm{d}t = \int_{t_0}^{t_1} \delta F \, \mathrm{d}t. \tag{III}$$

As before, the variation of the integral (10) represents approximately the increment of the integral when we pass from a given motion to a comparative motion. The difference between the true increment and the variation δI is "infinitesimal" in comparison with $|\delta x| + |\delta x^{\cdot}|$.

Variation of a derivative. Let the function

$$x = \Phi(q, t) \tag{12}$$

be given.

Let us assume that q is a certain function of the time t; hence

$$q = q(t) \qquad (t_0 \leq t \leq t_1). \tag{13}$$

Regarding q in formula (12) as an independent variable function, we obtain for the variation of the dependent variable x

$$\delta x = \frac{\partial x}{\partial q}\, \delta q, \tag{14}$$

where δq denotes an arbitrary function continuous together with its first and second derivatives in the interval $\langle t_0, t_1 \rangle$.

Let us form the derivative of (12). We get

$$x^{\cdot} = \frac{\partial x}{\partial q} q^{\cdot} + \frac{\partial x}{\partial t};$$

the variation $\delta x^{\cdot}$ of this function is therefore

$$\delta x^{\cdot} = \frac{\partial x^{\cdot}}{\partial q}\, \delta q + \frac{\partial x^{\cdot}}{\partial q^{\cdot}}\, \delta q^{\cdot}.$$

From (12) we see that the derivatives $\delta x \,/\, \delta q$ and $\delta x \,/\, \delta t$ do not depend on $q^{\cdot}$, because x does not depend on $q^{\cdot}$. Hence we obtain

$$\delta x^{\cdot} = \left(\frac{\partial^2 x}{\partial q^2} q^{\cdot} + \frac{\partial^2 x}{\partial q\, \partial t}\right) \delta q + \frac{\partial x}{\partial q}\, \delta q^{\cdot}. \tag{15}$$

Forming the derivative of (14) with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x) = \left(\frac{\partial^2 x}{\partial q^2} q^{\cdot} + \frac{\partial^2 x}{\partial q\, \partial t}\right) \delta q + \frac{\partial x}{\partial q}\, \delta q^{\cdot},$$

whence by (15)

$$\delta x^{\cdot} = \frac{\mathrm{d}(\delta x)}{\mathrm{d}t}. \tag{16}$$

Comparing (16) with formula (I) we see that both formulae have the same form. The difference lies in the fact that in formula (I) x is an independent variable and in (16) it is a dependent variable.

Formula (I) holds, therefore, regardless of whether x is a dependent or independent variable. It follows from this that *the variation is inter-*

changeable with the derivative, i. e. we obtain the same result by first forming the variation and then taking the derivative or conversely.

Variation of a compound function. Let the functions:

$$T = F(x, x^{\cdot}, t), \tag{17}$$

$$x = \Phi(q, t) \tag{18}$$

be given.

Let us assume that q is a function of the variable t:

$$q = q(t) \qquad (t_0 \leqq t \leqq t_1). \tag{19}$$

Forming the derivative of (18), we obtain

$$x^{\cdot} = \frac{\partial x}{\partial q} q^{\cdot} + \frac{\partial x}{\partial t}. \tag{20}$$

Substituting in (17) for x and $x^{\cdot}$ their values from (18) and (20), we obtain T as a function of the variables q, $q^{\cdot}$, t, i. e.

$$T = \Psi(q, q^{\cdot}, t), \tag{21}$$

and hence as its variation

$$\delta T = \frac{\partial T}{\partial q} \delta q + \frac{\partial T}{\partial q^{\cdot}} \delta q^{\cdot}. \tag{22}$$

From the theorem on the derivative of a compound function we obtain:

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q}, \qquad \frac{\partial T}{\partial q^{\cdot}} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial q^{\cdot}} + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q^{\cdot}}, \tag{23}$$

and by (18):

$$\frac{\partial x}{\partial q^{\cdot}} = 0. \tag{24}$$

Substituting (24) in (23) and then in (22), we obtain

$$\partial T = \left(\frac{\partial T}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q}\right)\delta q + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q^{\cdot}}\delta q^{\cdot} =$$

$$= \frac{\partial T}{\partial x}\left(\frac{\partial x}{\partial q}\delta q\right) + \frac{\partial T}{\partial x^{\cdot}}\left(\frac{\partial x^{\cdot}}{\partial q}\delta q + \frac{\partial x^{\cdot}}{\partial q^{\cdot}}\delta q^{\cdot}\right).$$

It is easy to verify that the expressions in the last two parentheses are variations of the functions (18) and (20), and therefore equal to δx and $\delta x^{\cdot}$. Consequently

$$\partial T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial x^{\cdot}}\delta x^{\cdot}. \tag{25}$$

Formula (25) represents the *variation of the compound function* (21), where δx and $\delta x^{\cdot}$ denote the variations of the functions (18) and (20). Let us note that (25) also represents the variation of function (17). We see from this that formula (25), i. e. formula (II), p. 505, holds regardless of whether x is the dependent or independent variable function.

Similarly, when t = const, the formula for the differential

$$\mathrm{d}T = \frac{\partial T}{\partial x}\,\mathrm{d}x + \frac{\partial T}{\partial x^{\cdot}}\,\mathrm{d}x^{\cdot}$$

holds regardless of whether x and $x^{\cdot}$ are dependent variables or not.

Let us note that $\delta x^{\cdot}$ in formula (25) is by (16) the derivative of δx.

Systems of points. Let us now define the variation in the case of a system of points.

Let there be given a system of n material points

$$A_1(x_1, y_1, z_1), \quad \ldots, \quad A_n(x_n, y_n, z_n).$$

Let us consider an arbitrary motion of the system (compatible with constraints or not) defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t),\ (t_0 \leqq t \leqq t_1,\ i = 1, 2, \ldots, n). \quad (26)$$

Let the function

$$T = F(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, y_1^{\cdot}, \ldots, y_n^{\cdot}, z_1^{\cdot}, \ldots, z_n^{\cdot}, t) \quad (27)$$

be given.

The *variation of function* (27) for a motion defined by functions (26) is given by the expression

$$\begin{aligned} \partial T = {} & \frac{\partial T}{\partial x_1}\,\delta x_1 + \ldots + \frac{\partial T}{\partial x_n}\,\delta x_n + \frac{\partial T}{\partial y_1}\,\delta y_1 + \ldots + \frac{\partial T}{\partial y_n}\,\delta y_n + \\ & + \frac{\partial T}{\partial z_1}\,\delta z_1 + \ldots + \frac{\partial T}{\partial z_n}\,\delta z_n + \frac{\partial T}{\partial x_1^{\cdot}}\,\delta x_1^{\cdot} + \ldots + \frac{\partial T}{\partial x_n^{\cdot}}\,\delta x_n^{\cdot} + \frac{\partial T}{\partial y_1^{\cdot}}\,\delta y_1^{\cdot} + \ldots + \\ & + \frac{\partial T}{\partial y_n^{\cdot}}\,\delta y_n^{\cdot} + \frac{\partial T}{\partial z_1^{\cdot}}\,\delta z_1^{\cdot} + \ldots + \frac{\partial T}{\partial z_n^{\cdot}}\,\delta z_n^{\cdot}, \end{aligned} \quad (28)$$

which we write more compactly as

$$\partial T = \sum_{i=1}^{n} \left(\frac{\partial T}{\partial x_i}\,\delta x_i + \frac{\partial T}{\partial y_i}\,\delta y_i + \frac{\partial T}{\partial z_i}\,\delta z_i + \frac{\partial T}{\partial x_i^{\cdot}}\,\delta x_i^{\cdot} + \frac{\partial T}{\partial y_i^{\cdot}}\,\delta y_i^{\cdot} + \frac{\partial T}{\partial z_i^{\cdot}}\,\delta z_i^{\cdot} \right), \quad \text{(IV)}$$

where δx_i, δy_i, δz_i are arbitrary functions continuous together with their first and second derivatives in the interval $\langle t_0, t_1 \rangle$, where

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x_i) = \delta x_i^{\cdot},\ \frac{\mathrm{d}}{\mathrm{d}t}(\delta y_i) = \delta y_i^{\cdot},\ \frac{\mathrm{d}}{\mathrm{d}t}(\delta z_i) = \delta z_i^{\cdot} \quad (i = 1, 2, \ldots, n). \quad \text{(V)}$$

The derivatives $\delta T / \partial x_1, \ldots, \delta T / \partial z_n^{\cdot}$ are partial derivatives of the functions (27), in which for $x_1, \ldots, z_n^{\cdot}$, are substituted the corresponding functions (26) and their derivatives.

As before, the variation δT denotes approximately the increment of the function T when we pass from the position of the system at the moment t in the given motion, to the position of the system at the same moment t in the comparative motion:

$$\mathbf{x}_i = x_i + \delta x_i,\ \mathbf{y}_i = y_i + \delta y_i,\ \mathbf{z}_i = z_i + \delta z_i \qquad (i = 1, 2, \ldots, n).$$

The difference between the true increment and the variation is — as is easily seen — "infinitesimal" in comparison with the sum

$$\sum_{i=1}^{n}(|\delta x_i| + |\delta y_i| + |\delta z_i| + |\delta x_i^{\cdot}| + |\delta y_i^{\cdot}| + |\delta z_i^{\cdot}|).$$

Let us note that the variation δT is obtained by forming the differential of the function T under the assumption that $t =$ const (i. e. for $\mathrm{d}t = 0$) and then writing δ instead of d.

Let there now be given an integral

$$I = \int_{t_0}^{t_1} T\,\mathrm{d}t \tag{29}$$

where T denotes the function (27).

The *variation of the integral* (29) for a motion defined by (26) is given by the expression

$$\delta I = \int_{t_0}^{t_1} \delta T\,\mathrm{d}t. \tag{30}$$

Therefore

$$\delta \int_{t_0}^{t_1} T\,\mathrm{d}t = \int_{t_0}^{t_1} \delta T\,\mathrm{d}t. \tag{VI}$$

Example 2. Determine the variation of the kinetic energy

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}).$$

We have:

$$\frac{\partial E}{\partial x_i} = 0,\ \frac{\partial E}{\partial y_i} = 0,\ \frac{\partial E}{\partial z_i} = 0,\ \frac{\partial E}{\partial x_i^{\cdot}} = m_i x_i^{\cdot},\ \frac{\partial E}{\partial y_i^{\cdot}} = m_i y_i^{\cdot},\ \frac{\partial E}{\partial z_i^{\cdot}} = m_i z_i^{\cdot},$$

consequently

$$\delta E = \Sigma m_i(x_i^{\cdot}\,\delta x_i^{\cdot} + y_i^{\cdot}\,\delta y_i^{\cdot} + z_i^{\cdot}\,\delta z_i^{\cdot}).$$

Example 3. Determine the variation of the function $\sqrt{T}$, where T is a function defined by formula (27).

Forming the differential under the assumption that $t = \text{const}$, we have

$$d\sqrt{T} = \tfrac{1}{2}dT / \sqrt{T}, \quad \text{whence} \quad \delta\sqrt{T} = \tfrac{1}{2}\delta T / \sqrt{T}.$$

Let us assume that the natural coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, are defined in terms of the parameters $q_1, \ldots, q_k$, by means of the functions:

$$x_i = f_i(q_1, \ldots, q_k, t), \; y_i = \varphi_i(q_1, \ldots, q_k, t), \; z_i = \psi_i(q_1, \ldots, q_k, t) \quad (i = 1, 2, \ldots, n). \tag{31}$$

We do not assume that the parameters are dependent nor that they are independent. Let

$$q_1 = q_1(t), \quad \ldots, \quad q_k = q_k(t) \qquad (t_0 \leqq t \leqq t_1) \tag{32}$$

be arbitrary functions continuous together with their first and second derivatives in the interval $\langle t_0, t_1 \rangle$. The functions (32) together with (33) define a certain motion of the system which may be compatible with the constraints or not. Differentiating (31), we obtain

$$x_i^{\cdot} = \frac{\partial x_i}{\partial q_1} q_1^{\cdot} + \ldots + \frac{\partial x_i}{\partial q_k} q_k^{\cdot} + \frac{\partial x_i}{\partial t}, \tag{33}$$

and similar formulae for $y_i^{\cdot}$ and $z_i^{\cdot}$. Let us substitute functions (31) for x_i, y_i, z_i, in (27), and functions (33) for $x_i^{\cdot}, y_i^{\cdot}, z_i^{\cdot}$. We obtain T in the form of a function of the variables $q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}, t$:

$$T = \mathbf{F}(q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}, t). \tag{34}$$

Proceeding as in the proof of the theorem on the variation of a compound function (*vide* formula (25), p. 508), it can be shown that the variation of function (34) is also expressed byf ormula (28) or (IV), where $\delta x_i, \delta y_i, \delta z_i$, are the variations of the functions (31), and $\delta x_i^{\cdot}, \delta y_i^{\cdot}, \delta z_i^{\cdot}$, the variations of functions (33) and of analogous ones for $y_i^{\cdot}, z_i^{\cdot}$.

Moreover, as in the proof of the theorem on the variation of a derivative (*vide* formula (16), p. 507) it can be proved that formulae (V), in which x_i, y_i, z_i, denote functions (31), will hold.

Let us form the variations of functions (31). We obtain:

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\delta q_k, \qquad \delta y_i = \frac{\partial y_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial y_i}{\partial q_k}\delta q_k,$$
$$\delta z_i = \frac{\partial z_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial z_i}{\partial q_1}\delta q_k. \tag{35}$$

Comparing (35) with formulae (III), p. 473, defining the virtual displacements, we see that they have the same formal appearance.

§ 2. Hamilton's principle. Actual motion. Let the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, act on the points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, of a system of n material points. Let us assume that the system is holonomic (without friction) and that the constraints are bilateral, defined by the relations:

$$F_j(x_1, y_1, z_1, \ldots, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m). \quad (1)$$

Let us consider an arbitrary system of functions

$$x_i = x_i(t), \quad y_i = y_i(t), \quad z_i = z_i(t) \qquad (i = 1, 2, \ldots, n) \quad (2)$$

continuous together with their first and second derivatives in the interval $\langle t_0, t_1 \rangle$. Functions (2) define a certain motion of the system.

If equations (1) are satisfied at each moment t when functions (2) are substituted for $x_1, \ldots, z_n$, then we say that functions (2) define the *motion* of the system *compatible with the constraints* or a *possible motion.*

The motion of the system which will actually take place under the action of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, is called the *actual motion.*

There can be a variety of actual motions, because this depends on the initial conditions.

Obviously an actual motion is always possible, because it satisfies relations (1). Conversely, however, not every possible motion is an actual motion.

For example, if a heavy point is constrained to remain constantly on a vertical line l (without friction), then the actual motion is a motion in which the acceleration is directed vertically downwards and equal in magnitude to the gravitational acceleration. On the other hand, a motion compatible with the constraints is every motion in which the point remains on the line l, in particular, a uniform motion as well as a motion in which the acceleration is not constant; these motions are obviously not actual motions.

From d'Alembert's principle it follows that among motions compatible with the constraints, only that motion is actual which satisfies at every moment the equation (II′), p. 475:

$$\sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\,\delta x_i + (P_{i_y} - m_i\ddot{y}_i)\,\delta y_i + (P_{i_z} - m_i\ddot{z}_i)\,\delta z_i] = 0, \quad (3)$$

where δx_i, δy_i, δz_i, are virtual displacements.

D'Alembert's principle therefore expresses a *characteristic property of actual motions,* distinguishing them from all motions compatible with the constraints.

Similarly, the equations of Lagrange (p. 481 and 486) and of Hamilton (p. 499) distinguish the actual motions from the set of all possible motions compatible with the constraints. However, in this chapter we shall meet with still other characteristic properties of actual motions expressed by means of integrals and variations. They are the so-called *integral variational principles.*

Comparative motion. Let us consider an arbitrary motion of a system compatible with the constraints, defined by functions (2), as well as the comparative motion:

$$x_i + \delta x_i,\ y_i + \delta y_i,\ z_i + \delta z_i \qquad (i = 1, 2, \dots, n). \quad (4)$$

Let us choose the variations $\delta x_i, \delta y_i, \delta z_i$, so that the variations of the functions (1) for the given motion (2) are zero:

$$\delta F_j = \frac{\partial F_j}{\partial x_1}\delta x_1 + \dots + \frac{\partial F_j}{\partial z_n}\delta z_n = 0 \qquad (j = 1, 2, \dots, m). \quad (5)$$

Comparing equations (5) with equations (I), p. 471, we see that δx_i, δy_i, δz_i, are at each moment the virtual displacements of the system.

If $\delta x_i, \delta y_i, \delta z_i$, are very small, then from (5) it follows that approximately

$$F_j(x_1 + \delta x_1, \dots, z_n + \delta z_n, t) = 0 \qquad (j = 1, 2, \dots, m), \quad (6)$$

i.e. that the comparative motion is approximately a motion compatible with the constraints. We express this by saying that the comparative motion (4) is compatible with the constraints for "infinitesimal" variations of $\delta x_i, \delta y_i, \delta z_i$, satisfying equations (5) (cf. p. 428).

Let us assume that the natural coordinates are defined in terms of the parameters $q_1, \dots, q_k$, by the functions:

$$x_i = f_i(q_1, \dots, q_k, t),\ y_i = \varphi_i(q_1, \dots, q_k, t),\ z_i = \psi_i(q_1, \dots, q_k, t) \quad (7)$$
$$(i = 1, 2, \dots, n).$$

Let us further assume that the parameters defining the position of the system compatible with the constraints must satisfy the relations:

$$\Phi_r(q_1, \dots, q_k, t) = 0 \qquad (r = 1, 2, \dots, s). \quad (8)$$

Let us consider an arbitrary system of functions continuous together with their first and second derivatives in the interval $\langle t_0, t_1\rangle$:

$$q_1 = q_1(t), \dots, q_k = q_k(t) \qquad (t_0 \leq t \leq t_1). \quad (9)$$

Let us assume, finally, that functions (8) become identically equal to zero when functions (9) are substituted for $q_1, \dots, q_k$.

Under these assumptions, substituting functions (9) in functions (7), we obtain functions of the time t defining a motion compatible with the constraints.

Let us consider a comparative motion $q_1 + \delta q_1, \ldots, q_k + \delta q_k$ and choose $\delta q_1, \ldots, \delta q_k$, such that for the given motion (9) the variations of the functions (8) are equal to zero:

$$\delta\Phi_r = \frac{\partial \Phi_r}{\partial q_1}\delta q_1 + \ldots + \frac{\partial \Phi_r}{\partial q_k}\delta q_k = 0 \qquad (r = 1, 2, \ldots, s). \tag{10}$$

Comparing (10) with formulae (IV), p. 473, we see that the variations $\delta q_1, \ldots, \delta q_k$, are virtual displacements at every moment t.

If $\delta q_1, \ldots, \delta q_k$, are very small, then by (10) we obtain approximately

$$\Phi_r(q_1 + \delta q_1, \ldots, q_k + \delta q_k, t) = 0 \qquad (r = 1, 2, \ldots, s). \tag{11}$$

Hence motion (11) will also be approximately compatible with the constraints. Using the same kind of expression as on p. 513, we can therefore say that if $\delta q_1, \ldots, \delta q_k$, are "infinitesimal" and satisfy equations (10), then the comparative motion is compatible with the constraints.

Hamilton's principle for natural coordinates. Let a system of n material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, of masses $m_1, \ldots, m_n$, be acted on by the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, depending on the variables $x_1, \ldots, z_n, x_1^{\cdot}, \ldots, z_n^{\cdot}, t$.

Therefore:

$$P_{i_x} = F_i(x_1, \ldots, z_n, x_1^{\cdot}, \ldots, z_n^{\cdot}, t), \; P_{i_y} = \Phi_i, \quad P_{i_z} = \Psi_i. \tag{12}$$

Let us assume that the system is holonomic (without friction) and that the constraints are bilateral. Let us consider the arbitrary functions:

$$x_i = x_i(t), \quad y_i = y_i(t), \quad z_i = z_i(t) \qquad (t_0 \leqq t \leqq t_1), \tag{13}$$

defining the motion of a system compatible with the constraints.

The kinetic energy of motion (13) is

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}). \tag{14}$$

Let us form the variation of the kinetic energy for the motion (13) (cf. example 2, p. 510):

$$\delta E = \sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i^{\cdot} + y_i^{\cdot}\,\delta y_i^{\cdot} + z_i^{\cdot}\,\delta z_i^{\cdot}). \tag{15}$$

But

$$x_i^{\cdot}\,\delta x_i^{\cdot} = x_i^{\cdot}\frac{\mathrm{d}(\delta x_i)}{\mathrm{d}t} = \frac{\mathrm{d}(x_i^{\cdot}\,\delta x_i)}{\mathrm{d}t} - x_i^{\cdot\cdot}\,\delta x_i \tag{16}$$

and similar formulae hold for $y_i^{\cdot}\,\delta y_i^{\cdot}$ and for $z_i^{\cdot}\,\delta z_i^{\cdot}$. Substituting these values in (15), we therefore get

$$\delta E = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i + y_i^{\cdot}\,\delta y_i + z_i^{\cdot}\,\delta z_i) - \sum_{i=1}^{n} m_i(x_i^{\cdot\cdot}\,\delta x_i + y_i^{\cdot\cdot}\,\delta y_i + z_i^{\cdot\cdot}\,\delta z_i). \tag{17}$$

Let us take

$$\delta' L = \sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i), \tag{18}$$

where P_{i_x}, P_{i_y}, P_{i_z}, denote functions (12), in which for x_i, y_i, z_i, and $x_i^{\cdot}, y_i^{\cdot}, z_i^{\cdot}$, the corresponding functions (13) and their derivatives have been substituted. From formulae (17) and (18) we obtain

$$\delta' L + \delta E = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i + y_i^{\cdot}\,\delta y_i + z_i^{\cdot}\,\delta z_i) + \\ + \sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i]. \tag{19}$$

Integrating both sides from t_0 to t_1, we obtain

$$\int_{t_0}^{t_1}[\delta' L + \delta E]\,\mathrm{d}t = \sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i + y_i^{\cdot}\,\delta y_i + z_i^{\cdot}\,\delta z_i)\Big|_{t_0}^{t_1} + \\ + \int_{t_0}^{t_1}\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i]\,\mathrm{d}t. \tag{20}$$

The symbol $|_{t_0}^{t_1}$ here means, as usual, that at first t_1 and then t_0 are to be substituted for t, and the resulting values subtracted from each other.

So far we have not used any principles of mechanics. Formula (20) therefore holds for an arbitrary motion (compatible with the constraints or not) defined by functions (13) if functions (12) are defined for this motion.

Let us now assume that motion (13) is an actual motion and that the variations δx_i, δy_i, δz_i, are virtual motions at every moment t.

Then from d'Alembert's principle it follows that at every moment t

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] = 0. \tag{21}$$

From formula (20) we obtain

$$\int_{t_0}^{t_1}(\delta' L + \delta E)\,\mathrm{d}t = \sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i + y_i^{\cdot}\,\delta y_i + z_i^{\cdot}\,\delta z_i)\Big|_{t_0}^{t_1}. \tag{22}$$

Let us assume, in addition, that the system is at the same position at t_0 and t_1 in motion (13) and in the comparative motion; i. e. that δx_i, δy_i, δz_i are zero for $t = t_0$ and $t = t_1$.

Under this assumption the right side of the equality (22) becomes zero and we obtain

$$\int_{t_0}^{t_1}(\delta' L + \delta E)\,\mathrm{d}t = 0. \tag{I}$$

Therefore: *equality* (I) *holds for an actual motion if* δx_i, δy_i, δz_i, *are virtual displacements at every moment and if they are zero for* $t = t_0$ *and* $t = t_1$.

This theorem is known as *Hamilton's principle.*

Hamilton's principle therefore gives a certain property of actual motions. We shall prove that *this property is characteristic*, i. e. that among the motions compatible with the constraints only actual motions satisfy Hamilton's principle. With this in view it is sufficient to prove that a motion compatible with the constraints and satisfying Hamilton's principle also satisfies d'Alembert's principle.

Proof. Let us assume that a motion defined by functions (13) and compatible with the constraints satisfies Hamilton's principle (I). If this did not satisfy d'Alembert's principle at a certain moment t' (where $t_0 < t' < t_1$), then it would be possible to find numbers $\overline{\delta x_i}$, $\overline{\delta y_i}$, $\overline{\delta z_i}$, defining the virtual displacement of the system at the moment t such that

$$\sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\,\overline{\delta x_i} + (P_{i_y} - m_i\ddot{y}_i)\,\overline{\delta y_i} + (P_{i_z} - m_i\ddot{z}_i)\,\overline{\delta z_i}] \neq 0 \tag{23}$$

for $t = t'$.

Let us choose the variations $\delta' x_i$, $\delta' y_i$, $\delta' z_i$, such that they define the virtual displacement of the system at each moment t and such that at the moment t'

$$\delta' x_i = \overline{\delta x_i}, \quad \delta' y_i = \overline{\delta y_i}, \quad \delta' z_i = \overline{\delta z_i};$$

from this by (23)

$$\sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\,\delta' x_i + (P_{i_y} - m_i\ddot{y}_i)\,\delta' y_i + (P_{i_z} - m_i\ddot{z}_i)\,\delta' z_i] = \\ = A \neq 0 \tag{24}$$

for $t = t'$.

Let us suppose, for instance, that $A > 0$ for $t = t'$. From the continuity of the motion it follows that in a certain small interval $\langle t', t''\rangle$ A is also greater than zero.

Consequently

$$A > 0, \quad \text{when} \quad t' \leqq t \leqq t''. \tag{25}$$

Let $\alpha(t)$ be an arbitrary function continuous together with its first and second derivatives in the interval $\langle t_0, t_1\rangle$, positive for $t' < t < t''$ and zero outside of this interval. Let us put:

$$\delta x_i = \alpha(t)\,\delta' x_i, \quad \delta y_i = \alpha(t)\,\delta' y_i, \quad \delta z_i = \alpha(t)\,\delta' z_i.$$

From this by (24) and (25)

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} [(P_{i_x} - m_i x_i^{\cdot\cdot})\, \delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\, \delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\, \delta z_i]\, \mathrm{d}t =$$

$$= \int_{t_0}^{t_1} A\, \alpha(t)\, \mathrm{d}t = \int_{t'}^{t''} A\, \alpha(t)\, \mathrm{d}t > 0. \tag{26}$$

Since the variations $\delta x_i, \delta y_i, \delta z_i$, represent the virtual displacements at every moment t and by assumption are equal to zero at $t = t_0$ and $t = t_1$, from formula (20) we therefore obtain by (26)

$$\int_{t_0}^{t_1} (\delta' L + \delta E)\, \mathrm{d}t > 0,$$

contrary to the assumption that the given motion satisfies Hamilton's principle.

In this manner we have proved that *the principles of d'Alembert and of Hamilton are equivalent.*

Example. A heavy point of mass m is constrained to remain on the sphere $x^2 + y^2 + z^2 - r^2 = 0$. We have (taking the z-axis directed vertically upwards):

$$\delta' L = -mg\, \delta z, \quad \delta E = m(x^\cdot\, \delta x^\cdot + y^\cdot\, \delta y^\cdot + z^\cdot\, \delta z^\cdot);$$

consequently by Hamilton's principle (I), p. 516,

$$\int_{t_0}^{t_1} [-g\, \delta z + x^\cdot\, \delta x^\cdot + y^\cdot\, \delta y^\cdot + z^\cdot\, \delta z^\cdot]\, \mathrm{d}t = 0.$$

This formula holds for an actual motion under the assumption that $\delta x, \delta y, \delta z$, are virtual displacements at every moment, i. e. that they satisfy the equation

$$x\, \delta x + y\, \delta y + z\, \delta z = 0,$$

and, in addition, become zero at $t = t_0$ and $t = t_1$.

Hamilton's principle for generalized coordinates. Let the natural coördinates be defined as functions of the parameters $q_1, \ldots, q_k$:

$$x_i = f_i(q_1, \ldots, q_k, t),\ y_i = \varphi_i(q_1, \ldots, q_k, t),\ z_i = \psi_i(q_1, \ldots, q_k, t) \tag{27}$$
$$(i = 1, 2, \ldots, n).$$

Let us assume that the parameters defining the positions of the system compatible with the constraints satisfy the equations

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s) \tag{28}$$

and let us consider an arbitrary actual motion of the system defined by the functions:

$$q_i = q_i(t) \qquad (i = 1, 2, \ldots, k). \tag{29}$$

The variations of the functions (27) for this motion are

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\,\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\,\delta q_k \tag{30}$$

and similarly for δy_i and δz_i.

Let us further assume that δq_i are virtual displacements at every moment t, i. e. that they satisfy equations (IV), p. 473:

$$\frac{\partial \Phi_r}{\partial q_1}\,\delta q_1 + \ldots + \frac{\partial \Phi_r}{\partial q_k}\,\delta q_k = 0 \qquad (r = 1, 2, \ldots, s); \tag{31}$$

consequently δx_i, δy_i, δz_i, are also virtual displacements (p. 473).

Finally, let us assume that δq_i are zero for $t = t_0$ and $t = t_1$; from (30) it follows that δx_i, δy_i, δz_i, will also be zero for $t = t_0$ and $t = t_1$. Since the variation is interchangeable with the derivative (p. 507), the variations of the first derivatives of the functions (27), i. e. $\delta x_i^{\cdot}$, $\delta y_i^{\cdot}$, $\delta z_i^{\cdot}$, are equal to the derivatives of the functions δx_i, δy_i, δz_i.

In virtue of (15), p. 514, and (18), p. 515, we can write Hamilton's principle (I), p. 515, in the form:

$$\int_{t_0}^{t_1}[\sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) + \sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i^{\cdot} + y_i^{\cdot}\,\delta y_i^{\cdot} + z_i^{\cdot}\,\delta z_i^{\cdot})]\,dt = 0, \tag{32}$$

where x_i, y_i, z_i, are functions defining the actual motion; δx_i, δy_i, δz_i, are the virtual displacements at every moment, assuming the value zero at $t = t_0$ and $t = t_1$; and $\delta x_i^{\cdot}$, $\delta y_i^{\cdot}$, $\delta z_i^{\cdot}$, are derivatives of the functions δx_i, δy_i, δz_i. As follows from the considerations of example 3, p. 510, equation (32) will also be satisfied if we assume that the functions x_i, y_i, z_i, given by the formulae (27) and (29), define the actual motion, while δx_i and $\delta x_i^{\cdot}$ are the variations of the functions (27) and their derivatives, where δq_i are virtual displacements equal to zero for $t = t_0$ and $t = t_1$.

Under these assumptions we have ((4), p. 483)

$$\delta' L = \sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) = \sum_{i=1}^{n} Q_j\,\delta q_j, \tag{33}$$

where Q_j denote the components of the generalized force. Moreover, from the theorem on the variation of a compound function (p. 508) we have

$$\sum_{i=1}^{n} m_i(x_i^{\cdot}\,\delta x_i^{\cdot} + y_i^{\cdot}\,\delta y_i^{\cdot} + z_i^{\cdot}\,\delta z_i^{\cdot}) = \delta(\tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}) = \delta E, \tag{34}$$

where the functions x_i, y_i, z_i, are given by formulae (27) and (29) defining the actual motion, $\delta x_i^{\cdot}$, $\delta y_i^{\cdot}$, $\delta z_i^{\cdot}$ are the variations of the derivatives of the functions (27), and E the kinetic energy expressed in terms of the parameters $q_1, \ldots, q_k$. By (32), (33) and (34) Hamilton's principle can therefore be written in the form (I), p. 515, where $\delta' L$ is defined by formula (33) and the kinetic energy E is expressed in terms of the generalized coordinates $q_1, \ldots, q_k$.

Therefore: *Hamilton's principle also holds for generalized coordinates under the assumption that δq_i are virtual displacements equal to zero for $t = t_0$ and $t = t_1$.*

Hamilton's principle in a potential field. Let us assume that a system of forces has a potential V. Consequently ((1), p. 434)

$$\delta' L = \delta V. \tag{35}$$

From Hamilton's principle we therefore obtain

$$\int_{t_0}^{t_1} [\delta V + \delta E]\, \mathrm{d}t = 0 \quad \text{or} \quad \int_{t_0}^{t_1} \delta(V + E)\, \mathrm{d}t = \delta \int_{t_0}^{t_1} (V + E)\, \mathrm{d}t = 0.$$

The expression $W = E + V$ was called the kinetic potential (p. 488). Hence

$$\delta \int_{t_0}^{t_1} W\, \mathrm{d}t = 0. \tag{II}$$

Therefore: *the variation of the integral of the kinetic potential is equal to zero for an actual motion if the variations $\delta x_i, \delta y_i, \delta z_i$, represent the virtual displacement of the system at every moment and if they are equal to zero at $t = t_0$ and $t = t_1$.*

Formula (35) holds for generalized coordinates (cf. (39), 463). Since Hamilton's principle also holds for generalized coordinates, (II) is satisfied for an actual motion under the assumption that the kinetic potential W is expressed in terms of the parameters $q_1, \ldots, q_k$, and the variation was formed for an actual motion, where δq_i are virtual displacements equal to zero at $t = t_0$ and $t = t_1$.

Holonomo-scleronomic systems in a potential field. Let a holonomo-scleronomic system be given in which the forces have a potential

$$V = V(x_1, \ldots, z_n). \tag{36}$$

Let us assume that the motion of a system defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t)\ \ (t_0 \leqq t \leqq t_1;\ i = 1, 2, \ldots, n) \tag{37}$$

is an actual motion for which the kinetic energy in $\langle t_0, t_1 \rangle$ does not vanish, i. e.:

$$E \neq 0 \quad \text{for} \quad t_0 \leqq t \leqq t_1. \tag{38}$$

By Hamilton's principle (I), p. 515, and (35) we have

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\,\delta x_i+\frac{\partial V}{\partial y_i}\,\delta y_i+\frac{\partial V}{\partial z_i}\,\delta z_i\right)\right]dt+$$
$$+\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}m_i(\dot{x}_i\,\delta\dot{x}_i+\dot{y}_i\,\delta\dot{y}_i+\dot{z}_i\,\delta\dot{z}_i)\right]dt=0, \tag{39}$$

where δx_i, δy_i, δz_i, are the virtual displacements at every moment and are equal to zero at $t = t_0$ and $t = t_1$. Let

$$t=\vartheta(\tau) \qquad (\tau_0 \leqq \tau \leqq \tau_1) \tag{40}$$

be an arbitrary function of the variable τ, continuous together with its first and second derivatives in the interval $\langle\tau_0, \tau_1\rangle$ and satisfying the conditions:

$$\vartheta(\tau_0)=t_0,\quad \vartheta(\tau_1)=t_1,\quad \vartheta'(\tau)<0 \qquad (\tau_0 \leqq \tau \leqq \tau_1), \tag{41}$$

where ϑ' denotes the derivative with respect to τ. Substituting (40) in (37), we obtain:

$$x_i = x_i(\vartheta(\tau)) = \xi_i(\tau),\quad y_i=\eta_i(\tau),\quad z_i=\zeta_i(\tau), \tag{42}$$
$$(\tau_0 \leqq \tau \leqq \tau_i;\ i=1, 2, \ldots, n).$$

Functions (42) represent the parametric equations of the path of the points of the system in terms of the parameter τ.

Denoting by x_i', y_i', z_i', the derivatives of the functions (42), and by t' the derivative of function (40) with respect to the parameter τ, we obtain:

$$\dot{x}_i = x_i'/t',\quad \dot{y}_i = y_i'/t',\quad \dot{z}_i = z_i'/t', \tag{43}$$

whence for the kinetic energy

$$E=\tfrac{1}{2}\sum_{i=1}^{n}m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)=\tfrac{1}{2}\sum_{i=1}^{n}\frac{m_i(x_i'^2+y_i'^2+z_i'^2)}{t'^2}.$$

From the principle of conservation of total energy we have

$$E-V=h,\quad \text{where}\quad h=\text{const}, \tag{45}$$

whence by (44)

$$t'=\sqrt{\frac{\tfrac{1}{2}\sum_{i=1}^{n}m_i(x_i'^2+y_i'^2+z_i'^2)}{h+V}}. \tag{46}$$

Formula (46) holds, because by (45) $h + V = E$ and by (38) $E \neq 0$.

Let us assume that the coordinates x_i, y_i, z_i, appearing in V (cf. formula (36)) are expressed in formula (46) by functions (42). In virtue of (40) the variations δx_i, δy_i, δz_i, which are functions of the variable t, can be regarded as functions of the variable τ.

Denoting by $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, the derivatives with respect to τ, we obtain:

$$\delta \dot{x}_i = \delta x_i' \,/\, t', \quad \delta \dot{y}_i = \delta y_i' \,/\, t', \quad \delta \dot{z}_i = \delta z_i' \,/\, t'. \tag{47}$$

Expressing the variable t in (39) in terms of τ by means of function (40), we get by (41), (43), and (47),

$$\int_{\tau_0}^{\tau_1} \left[\sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right)\right] t' \mathrm{d}\tau + \\ + \int_{\tau_0}^{\tau_1} \left[\sum_{i=1}^{n} \frac{m_i(x_i'\,\delta x_i' + y_i'\,\delta y_i' + z_i'\,\delta z_i')}{t'^2}\right] t'\,\mathrm{d}\tau = 0, \tag{48}$$

where x_i, y_i, z_i, δx_i, δy_i, δz_i, are functions of the variable τ. Formula (48) can be written in the form

$$\int_{\tau_0}^{\tau_1} \left[\delta V \cdot t' + \frac{1}{t'}\,\delta\,\tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2)\right] \mathrm{d}\tau = 0, \tag{49}$$

where the concept of variation is to be understood as before, but now τ appears instead of t. Substituting (46) in (49), we obtain

$$\int_{\tau_0}^{\tau_1} \left[\frac{\delta V}{\sqrt{h+V}} \sqrt{\tfrac{1}{2}\Sigma m_i(x_i'^2 + y_i'^2 + z_i'^2)}\right] \mathrm{d}\tau + \\ \int_{\tau_0}^{\tau_1} \left[+\sqrt{h+V} \cdot \frac{\delta \Sigma \frac{1}{2} m_i(x_i'^2 + y_i'^2 + z_i'^2)}{\sqrt{\Sigma \frac{1}{2} m_i(x_i'^2 + y_i'^2 + z_i'^2)}}\right] \mathrm{d}\tau = 0. \tag{50}$$

It is easy to verify that the integrand is equal to the variation of

$$2\delta \left[\sqrt{h+V} \sqrt{\tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2)}\right],$$

whence by (50)

$$\delta \int_{\tau_0}^{\tau_1} \left[\sqrt{h+V} \sqrt{\sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2)}\right] \mathrm{d}\tau = 0. \tag{I}$$

From the assumptions concerning δx_i, δy_i, δz_i, it follows that formula (I) holds for arbitrary functions δx_i, δy_i, δz_i, of the parameter τ, which for every value τ are the virtual displacements in the position of the system defined by functions (42), and which are zero for $\tau = \tau_0$ and $\tau = \tau_1$. The variation is formed for the functions (42), which define the path of the points of the system in an actual motion.

Since the time t does not appear in formula (I), this formula expresses a certain property of the paths of the actual motion.

In particular, let the system consist of one point x, y, z, of mass m, moving without the action of forces. Consequently $V = 0$. From (I) we obtain

$$\delta\int_{\tau_0}^{\tau_1}\sqrt{x'^2 + y'^2 + z'^2}\,\mathrm{d}\tau = 0.$$

Since the differential of arc is $\mathrm{d}s = \sqrt{x'^2 + y'^2 + z'^2}\,\mathrm{d}\tau$,

$$\delta\int_{\tau_0}^{\tau_1}\mathrm{d}s = 0. \tag{51}$$

Let us assume that the point is constrained to the surface S.

Curves having property (51) are the so-called *geodesics*. In differential geometry it is proved that the shortest curve on a surface joining two sufficiently close points of the geodesic is an arc of this geodesic. From (51) it follows, therefore, that *the motion of a point on a surface without the action of forces always takes place along geodesics.*

Remark. Formula (I) is usually proved more easily from the so-called *principle of Maupertuis* (p. 530). However, it is then necessary to make use of certain theorems from the calculus of variations which we do not assume to be known to the reader.

§ 3. Variation with the variation of time. Variation of a function. Let a motion along the x-axis be defined by the function

$$x = x(t) \qquad (t_0 \leqq t \leqq t_1). \tag{1}$$

Let us consider an arbitrary comparative motion

$$\mathbf{x} = \mathbf{x}(t) \qquad (t_0' \leqq t \leqq t_1'), \tag{2}$$

where the moments t_1' and t_0' can be different from t_0 and t_1. Let us denote by Δt and arbitrary function of the time t, continuous together with its first and second derivatives in the interval $\langle t_0, t_1\rangle$ and satisfying the inequality

$$t_0' \leqq t + \Delta t \leqq t_1' \qquad (t_0 \leqq t \leqq t_1). \tag{3}$$

Finally, let

$$\Delta x = \mathbf{x}(t + \Delta t) - x(t); \tag{I}$$

Δx is therefore a function of the time t and denotes the increment of the coordinate x, when we pass from the point A in the given motion at the moment t to the point A' in thecomparative motion at the moment $t + \Delta t$. Forming the derivative of (I), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Delta x) = [\mathbf{x}^{\cdot}(t + \Delta t)]\left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right) - x^{\cdot}(t),$$

whence

$$\mathbf{x}^{\cdot}(t + \Delta t) = \left[\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} + x^{\cdot}(t)\right] \Big/ \left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right).$$

Subtracting $x^{\cdot}(t)$ from both sides, we get after some easy transformations

$$\mathbf{x}^{\cdot}(t + \Delta t) - x^{\cdot}(t) = \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} - x^{\cdot}(t)\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t} + \varepsilon, \tag{4}$$

where

$$\varepsilon = \eta \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}, \quad \eta = -\left(\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} - x^{\cdot}(t)\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right) \Big/ \left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right). \tag{5}$$

Therefore, if Δx, Δt, and their derivatives, tend to zero uniformly, then η tends to zero uniformly. Consequently R is "infinitesimal" in comparison with $|\mathrm{d}(\Delta x) / \mathrm{d}t| + |\mathrm{d}(\Delta t) / \mathrm{d}t|$.

Let

$$\Delta x^{\cdot} = \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} - x^{\cdot}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{II}$$

Therefore by (4)

$$\mathbf{x}^{\cdot}(t + \Delta t) - x^{\cdot}(t) = \Delta x^{\cdot} + \varepsilon. \tag{6}$$

The left side of equality (6) denotes the increment of the velocity of the points A and A', i. e. the increment of the velocity when we pass from the point A at the moment t in the given motion to the point A' at the moment $t + \Delta t$ in the comparative motion. Consequently $\Delta x^{\cdot}$ represents this increment approximately, with a difference which is "infinitesimal" as compared with the sum $|\mathrm{d}(\Delta x) / \mathrm{d}t| + |\mathrm{d}(\Delta t) / \mathrm{d}t|$.

Let us note that by (6) we have in general

$$\Delta x^{\cdot} \neq \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t}. \tag{II'}$$

Let the function

$$T = F(x, x^{\cdot}, t) \tag{7}$$

be given.

Let us denote by T the value of the function (7) in a given motion at the moment t, and by $\mathbf{T}$ the value of this function in a comparative motion at the moment $t + \Delta t$. Consequently the difference of these values is

$$\mathbf{T} - T = F(\mathbf{x}(t + \Delta t), \mathbf{x}^{\cdot}(t + \Delta t), t + \Delta t) - F(x(t), x^{\cdot}(t), t). \tag{8}$$

From Taylor's formula we obtain

$$\mathbf{T} - T = \frac{\partial T}{\partial x}(\mathbf{x}(t + \Delta t) - x(t)) + \frac{\partial T}{\partial x^{\cdot}}(\mathbf{x}^{\cdot}(t + \Delta t) - x^{\cdot}(t)) + \frac{\partial T}{\partial t}\Delta t + R, \tag{9}$$

where the remainder R is an "infinitesimal" of higher order than the increments $\mathbf{x}(t+\Delta t) - x(t)$, $\mathbf{x}^{\cdot}(t+\Delta t) - x^{\cdot}(t)$ and Δt.

In virtue of (I), (6), and (9), we obtain

$$\mathbf{T} - T = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial x^{\cdot}}\Delta x^{\cdot} + \frac{\partial T}{\partial t}\Delta t + R', \tag{10}$$

where

$$R' = R + \varepsilon \frac{\partial T}{\partial x^{\cdot}},$$

and hence where R' is "infinitesimal" as compared with

$$|\Delta x| + |\Delta t| + |\mathrm{d}(\Delta x) / \mathrm{d}t| + |\mathrm{d}(\Delta t) / \mathrm{d}t|. \tag{11}$$

Let us put

$$\Delta T = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial x^{\cdot}}\Delta x^{\cdot} + \frac{\partial T}{\partial t}\Delta t. \tag{III}$$

By (10) we have $\mathbf{T} - T = \Delta T + R'$; consequently ΔT denotes approximately the increment of the function T when we pass from the point A at the moment t in the given motion to the point A' at the moment $t + \Delta t$ in the comparative motion, where the error committed is "infinitesimal" as compared with (11).

The expression ΔT is called the *variation together with the variation of time* of the function T for the motion $x = x(t)$, and the function Δt is called the *variation of time*.

In virtue of (I), p. 505, and (5), p. 504, we shall have $\Delta x = \delta x$ for $\Delta t = 0$, and by (II) and (I), p. 505, $\Delta x^{\cdot} = \delta x^{\cdot}$. From (III) we therefore get

$$\Delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial x^{\cdot}}\delta x^{\cdot},$$

i. e. $\Delta T = \delta T$.

Hence, when $\Delta t = 0$, the variation together with the variation of time becomes an ordinary variation.

Example. Form the variation together with the variation of time for the function

$$T = \tfrac{1}{2}mx^{\cdot 2} + \alpha x t,$$

where α and m are constants.

We have

$$\Delta T = \alpha t\, \Delta x + m x^{\cdot}\, \Delta x^{\cdot} + \alpha x\, \Delta t;$$

hence by (II)

$$\Delta T = \alpha t\, \Delta x + m x^{\cdot} \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} + \alpha x\, \Delta t - m x^{\cdot 2} \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}.$$

Systems of points. Let the motion of a system of n material points be defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t),\ (t_0 \leqq t \leqq t_1;\ i = 1, 2, \ldots, n) \quad (12)$$

and let the function

$$T = F(x_1, \ldots, z_n, x_1^{\cdot}, \ldots, z_n^{\cdot}, t) \quad (13)$$

be given.

Let us consider an arbitrary comparative motion

$$\mathbf{x}_i = \mathbf{x}_i(t),\ \mathbf{y}_i = \mathbf{y}_i(t),\ \mathbf{z}_i = \mathbf{z}_i(t),\ (t_0' \leqq t \leqq t_1';\ i = 1, 2, \ldots, n). \quad (14)$$

Let Δt be an arbitrary function of the time t, continuous with its first and second derivatives in the interval $\langle t_0, t_1 \rangle$ and satisfying the condition

$$t_0' \leqq t + \Delta t \leqq t_1' \qquad (t_0 \leqq t \leqq t_1). \quad (15)$$

Let us put:

$$\Delta x_i = \mathbf{x}_i(t + \Delta t) - x_i(t),\quad \Delta y_i = \mathbf{y}_i(t + \Delta t) - y_i(t),\quad \Delta z_i = \mathbf{z}_i(t + \Delta t) - z_i(t), \quad \text{(IV)}$$

$$\Delta x_i^{\cdot} = \frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} - x_i^{\cdot} \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t},\quad \Delta y_i^{\cdot} = \frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} - y_i^{\cdot} \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t},\quad \Delta z_i^{\cdot} = \frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t} - z_i^{\cdot} \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \quad \text{(V)}$$

The expressions Δx_i, Δy_i, Δz_i, denote the increments of the coordinates, and $\Delta x_i^{\cdot}$, $\Delta y_i^{\cdot}$, $\Delta z_i^{\cdot}$, the approximate increments of the derivatives of these coordinates when we pass from the position of the system at the moment t in the given motion to the position of the system at the moment $t + \Delta t$ in the comparative motion. The error made by this approximation is "infinitesimal" as compared with

$$\left|\frac{\mathrm{d}(\Delta x_1)}{\mathrm{d}t}\right| + \left|\frac{\mathrm{d}(\Delta y_1)}{\mathrm{d}t}\right| + \ldots + \left|\frac{\mathrm{d}(\Delta z_n)}{\mathrm{d}t}\right| + \left|\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right|. \quad (16)$$

Let us denote by T the value of the function (13) in the given motion at the moment t, and by $\mathbf{T}$ its value at the moment $t + \Delta t$ in the comparative motion. Putting

$$\Delta T = \sum_{i=1}^{n} \left(\frac{\partial T}{\partial x_i} \Delta x_i + \frac{\partial T}{\partial y_i} \Delta y_i + \frac{\partial T}{\partial z_i} \Delta z_i \right) + \\ + \sum_{i=1}^{n} \left(\frac{\partial T}{\partial \dot{x}_i} \Delta \dot{x}_i + \frac{\partial T}{\partial \dot{y}_i} \Delta \dot{y}_i + \frac{\partial T}{\partial \dot{z}_i} \Delta \dot{z}_i \right) + \frac{\partial T}{\partial t} \Delta t \qquad \text{(VI)}$$

and proceeding as before, we obtain

$$\mathbf{T} - T = \Delta T + R, \qquad (17)$$

where R is "infinitesimal" as compared with the sum

$$\sum_{i=1}^{n} \left(|\Delta x_i| + |\Delta y_i| + |\Delta z_i| + \left| \frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} \right| + \left| \frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} \right| + \left| \frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t} \right| \right) + \\ + |\Delta t| + \left| \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t} \right|. \qquad (18)$$

The expression ΔT is called the *variation together with the variation of time* of the function T for the motion (12) of a system of points.

The variation ΔT therefore represents approximately the increment of the function T when we pass from the position of the system at the moment t in a given motion to the position of a system at the moment $t + \Delta t$ in a comparative motion, where the difference between the true increment and ΔT is "infinitesimal" as compared with (18).

In virtue of (IV), (V), (5), p. 504, and (I), p. 505, we get for $\Delta t = 0$:

$$\Delta x_i = \delta x_i,\ \Delta y_i = \delta y_i,\ \Delta z_i = \delta z_i,\ \Delta \dot{x}_i = \delta \dot{x}_i,\ \Delta \dot{y}_i = \delta \dot{y}_i,\ \Delta \dot{z}_i = \delta \dot{z}_i,$$

whence by (VI) $\Delta T = \delta T$.

Hence, when $\Delta t = 0$, the variation together with the variation of time becomes an ordinary variation.

Variation together with the variation of time of an integral. Let the integral

$$I = \int_{t_0}^{t_1} T \,\mathrm{d}t \qquad (19)$$

be given, where T denotes the function (13). Let us denote by Δt_0 and Δt_1 the values of the function Δt at t_0 and t_1.

Let I be the value of the integral (19) for a given motion, and $\mathbf{I}$ the value of this integral for a comparative motion taken between the limits $t_0 + \Delta t_0$ and $t_1 + \Delta t_1$, i. e.

$$\mathbf{I} = \int_{t_0+\Delta t_0}^{t_1+\Delta t_1} T_1 \,\mathrm{d}t, \qquad (20)$$

where T_1 denotes the value of the function T at the moment t in the comparative motion. Substituting $t + \Delta t$ for t in (20), we obtain

$$\mathbf{I} = \int_{t_0}^{t_1} \mathbf{T}\left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right) \mathrm{d}t, \tag{21}$$

where $\mathbf{T}$ denotes the value of the function T at the moment $t + \Delta t$ in the comparative motion.

By (19) and (21) we obtain

$$\mathbf{I} - I = \int_{t_0}^{t_1} \left[(\mathbf{T} - T) + \mathbf{T}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right] \mathrm{d}t,$$

whence by (17) after some easy transformations

$$\mathbf{I} - I = \int_{t_0}^{t_1} \left[\Delta T + T\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right] \mathrm{d}t + R', \tag{22}$$

where

$$R' = \int_{t_0}^{t_1} \left[(\Delta T + R)\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t} + R\right] \mathrm{d}t. \tag{23}$$

It is easy to verify that R' is "infinitesimal" in comparison with (18). Let us put

$$\Delta I = \Delta \int_{t_0}^{t_1} T\,\mathrm{d}t = \int_{t_0}^{t_1} \left[\Delta T + T\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right] \mathrm{d}t. \tag{VII}$$

The expression ΔI is called the *variation together with the variation of time of the integral I.*

By (22) and (VII) we have

$$\mathbf{I} - I = \Delta I + R', \tag{24}$$

ΔI therefore represents approximately the increment of the integral (19) when we pass from the given motion to the comparative motion; we calculate the integral between the limits $t_0 + \Delta t_0$, $t_1 + \Delta t_1$ in the comparative motion. The difference between ΔI and the true increment is "infinitesimal" in comparison with (18).

In the case when $\Delta t = 0$, the variation together with the variation of time becomes — as it is easily seen — an ordinary variation.

§ 4. Maupertuis' principle (of least action). Hölder's transformation. Let a system of material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, of masses $m_1, \ldots, m_n$, be subjected to the action of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, depending on $x_1, \ldots, z_n, x_1^{\cdot}, \ldots, z_n^{\cdot}, t$. Therefore

$$P_{i_x} = F_i(x_1, \ldots, z_n, x_1^{\cdot}, \ldots, z_n^{\cdot}, t), \quad P_{i_y} = \Phi_i,\ P_{i_z} = \Psi_i. \tag{1}$$

Let us assume that the system is holonomic without friction and that the constraints are bilateral.

Let us consider an arbitrary motion of the system compatible with the constraints or not, defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t)\ (t_0 \leqq t \leqq t_1;\ i = 1, 2, \ldots, n). \tag{2}$$

The kinetic energy is

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \tag{3}$$

Let us form the variation of the kinetic energy together with the variation of time for the motion (2):

$$\Delta E = \sum_{i=1}^{n} m_i(\dot{x}_i\,\Delta\dot{x}_i + \dot{y}_i\,\Delta\dot{y}_i + \dot{z}_i\,\Delta\dot{z}_i). \tag{4}$$

Expressing $\Delta\dot{x}_i$, $\Delta\dot{y}_i$, $\Delta\dot{z}_i$, by means of formulae (V), p. 525, we get

$$\Delta E = \sum_{i=1}^{n} m_i\left[\dot{x}_i\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right] - 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{5}$$

Transposing the last term to the left and integrating, we obtain

$$\int_{t_0}^{t_1}\left[\Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \int_{t_0}^{t_1}\sum_{i=1}^{n} m_i\left[\dot{x}_i\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right]\mathrm{d}t. \tag{6}$$

Integrating by parts, we obtain

$$\int_{t_0}^{t_1}\dot{x}_i\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} = \dot{x}_i\,\Delta x_i\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\ddot{x}_i\,\Delta x_i\,\mathrm{d}t$$

and similar formulae are obtained for y_i and z_i. Applying them to the right side of equation (6), we get

$$\int_{t_0}^{t_1}\left[\Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \sum_{i=1}^{n} m_i(\dot{x}_i\,\Delta x_i + \dot{y}_i\,\Delta y_i + \dot{z}_i\,\Delta z_i)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\Big[\sum_{i=1}^{n} m_i(\ddot{x}_i\,\Delta x_i + \ddot{y}_i\,\Delta y_i + \ddot{z}_i\,\Delta z_i)\Big]\,\mathrm{d}t. \tag{7}$$

Let us put

$$\Delta' L = \sum_{i=1}^{n}(P_{i_x}\,\Delta x_i + P_{i_y}\,\Delta y_i + P_{i_z}\,\Delta z_i). \tag{8}$$

Integrating formula (8) and adding to both sides of equation (7), we get

$$\int_{t_0}^{t_1}\left[\Delta' L + \Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \sum_{i=1}^{n} m_i(x_i^{\cdot}\,\Delta x_i + y_i^{\cdot}\,\Delta y_i + z_i^{\cdot}\,\Delta z_i)\Big|_{t_0}^{t_1} +$$

$$+\int_{t_0}^{t_1}\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\Delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\Delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\Delta z_i]\,\mathrm{d}t. \quad \text{(I)}$$

Formula (I) is called *Hölder's transformation.*

It holds for every motion, whether compatible with the constraints or not (on contition that the functions (1) are defined for this motion).

If we take the ordinary variation δ instead of the variation Δ together with the variation of time, i. e. if we put $\Delta t = 0$, then — as it is easily seen — we obtain formula (20), p. 515.

More general form of Hamilton's principle. Let us assume that functions (2) define an actual motion. In addition, let us assume that the functions $\Delta x_i, \Delta y_i, \Delta z_i$, are virtual displacements of the system at every moment t.

By d'Alembert's principle the integrand on the right side of (I) is zero. Consequently

$$\int_{t_0}^{t_1}\left[\Delta' L + \Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \sum_{i=1}^{n} m_i(x_i^{\cdot}\,\Delta x_i + y_i^{\cdot}\,\Delta y_i + z_i^{\cdot}\,\Delta z_i)\Big|_{t_0}^{t_1}.$$

Let us assume that Δx_i, Δy_i, Δz_i, are equal to zero at $t = t_0$ and $t = t_1$. The right side of the last equality will therefore be zero. Hence we obtain

$$\int_{t_0}^{t_1}\left[\Delta' L + \Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = 0. \quad \text{(II)}$$

Equation (II) *holds for an actual motion under the assumption that* Δx_i, Δy_i, Δz_i, *are virtual displacements at every moment* t *and are equal to zero for* $t = t_0$ *and* $t = t_1$, *while* Δt *is arbitrary.*

When $\Delta t = 0$ the variation Δ becomes the variation δ. It is easy to see that (II) then assumes the form of Hamilton's principle (I), p. 515. Form (II) of the variational principle is therefore more general than Hamilton's principle. However, it does not represent a more general property. For by (5) and (8) we can write (II) in the form

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}(P_{i_x}\,\Delta x_i + P_{i_y}\,\Delta y_i + P_{i_z}\,\Delta z_i)\right]\mathrm{d}t +$$

$$+\int_{t_0}^{t_1}\left[\sum_{i=1}^{n} m_i\left(\dot{x}_i\,\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i\,\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i\,\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right)\right]\mathrm{d}t = 0. \tag{9}$$

Since Δx_i, Δy_i, Δz_i, are arbitrary functions representing virtual displacements and vanishing at $t = t_0$ and $t = t_1$, while Δt does not appear at all in formula (9), writing δx_i, δy_i, δz_i, for Δx_i, Δy_i, Δz_i, we obtain Hamilton's principle from (9).

Equation (II) is therefore equivalent to Hamilton's principle.

Maupertuis' principle. Equation (II) holds for an arbitrary Δt, while Δx_i, Δy_i, Δz_i, should only be virtual displacements at every moment t, vanishing at $t = t_0$ and $t = t_1$.

Let us now assume that Δx_i, Δy_i, Δz_i, and Δt are so chosen that they satisfy, in addition, the condition

$$\Delta' L = \Delta E. \tag{III}$$

By (5) and (8) condition (III) can be written in the form

$$\sum_{i=1}^{n}(P_{i_x}\,\Delta x_i + P_{i_y}\,\Delta y_i + P_{i_z}\,\Delta z_i) =$$

$$= \sum_{i=1}^{n} m_i\left(\dot{x}_i\,\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i\,\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i\,\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right) - 2E\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{10}$$

From (II) and (III) we obtain

$$\int_{t_0}^{t_1}\left[2\Delta E + 2E\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right] = 0, \tag{11}$$

whence by formula (VII), p. 527,

$$\Delta\int_{t_0}^{t_1} E\,\mathrm{d}t = 0. \tag{IV}$$

Therefore: *the variation together with the variation of time of the integral of the kinetic energy is zero for an actual motion if* Δx_i, Δy_i, Δz_i, *are virtual displacements at every moment, equal to zero at* $t = t_0$ *and* $t = t_1$, *and if condition* (III) *holds* (i. e. if the virtual work for the displacement Δx_i, Δy_i, Δz_i, is equal to the variation together with the variation of time of the kinetic energy).

This theorem is called *Maupertuis' principle* or the *principle of the least action.*

Denoting by $\mathrm{d}s_i$ the differential of arc along which the point m_i moves, and by v_i the velocity of the point, we have $\mathrm{d}s_i = v_i\,\mathrm{d}t$. Consequently

$$\int_{t_0}^{t_1} E\,\mathrm{d}t = \int_{t_0}^{t_1} (\tfrac{1}{2}\sum_{i=1}^{n} m_i v_i^2)\,\mathrm{d}t = \tfrac{1}{2}\int_{t_0}^{t_1} \sum_{i=1}^{n} m_i v_i\,\mathrm{d}s_i.$$

The expression $m_i v_i$ is the *momentum* ("*action*").

On the basis of (IV) it can be proved that under certain assumptions the integral of the kinetic energy for an actual motion has the smallest value among motions satisfying certain conditions. Hence the name *principle of the least action.*

Let us assume that a certain motion compatible with the constraints satisfies Maupertuis' principle, and, in addition, that the kinetic energy does not vanish in $\langle t_0, t_1\rangle$:

$$E \neq 0 \qquad (t_0 \leqq t \leqq t_1). \quad (12)$$

Let the functions $\Delta x_i, \Delta y_i, \Delta z_i$, at every moment t be virtual displacements, equal to zero at $t = t_0$ and $t = t_1$, and arbitrary in other respects. Let us choose Δt so that equality (III), or — which amounts to the same thing — equation (10), holds. In virtue of (12) and (10) we can assume

$$\Delta t = \int_{t_0}^{t_1} \frac{1}{2E}\left[\sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right) - \right.$$
$$\left. - \sum_{i=1}^{n}(P_{i_x}\Delta x_i + P_{i_y}\Delta y_i + P_{i_z}\Delta z_i)\right]\mathrm{d}t. \quad (13)$$

Formula (IV) holds for Δx_i, Δy_i, Δz_i, Δt, chosen in the above way, and consequently (11) also holds. By (III) and (11)

$$\int_{t_0}^{t_1}\left[2\Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \int_{t_0}^{t_1}\left[\Delta' L + \left(\Delta' L + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right)\right]\mathrm{d}t = 0,$$

whence by (10) we obtain formula (9), in which $\Delta x_i, \Delta y_i, \Delta z_i$, satisfy the same conditions as $\delta x_i, \delta y_i, \delta z_i$, in Hamilton's principle. Since, as we have proved (p. 529), (9) is equivalent to Hamilton's principle, the given motion is an actual motion. Hence we see that among these motions compatible with the constraints for which $E \neq 0$, only actual motions satisfy Maupertuis' principle.

Therefore: *Maupertuis' principle represents a characteristic property of those actual motions for which* $E \neq 0$.

Let us assume that a motion takes place in a potential field having a potential V. Consequently:

$$\partial V / \partial x_i = P_{i_x}, \quad \partial V / \partial y_i = P_{i_y}, \quad \partial V / \partial z_i = P_{i_z},$$

$$\Delta' L = \sum_{i=1}^{n}(P_{i_x}\Delta x_i + P_{i_y}\Delta y_i + P_{i_z}\Delta z_i) = $$
$$= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\Delta x_i + \frac{\partial V}{\partial y_i}\Delta y_i + \frac{\partial V}{\partial z_i}\Delta z_i\right).$$

Since V is a function of the coordinates x_i, y_i, z_i, only, it follows that $\Delta' L = \Delta V$, and hence condition (III) can be written in the form $\Delta V = \Delta E$; consequently

$$\Delta(E - V) = 0. \tag{III'}$$

Remark 1. The assumption $E \neq 0$ is essential; this means that if a motion compatible with the constraints satisfies Maupertuis' principle but not the condition $E \neq 0$, then the motion need not be an actual motion.

For example, let some scleronomic system be given. Let us consider a motion in which the system is at rest from t_0 to t_1 in a certain position compatible with the constraints. Therefore we have $E = 0$ constantly, whence by (4) $\Delta E = 0$ constantly also. It follows from this that formula (11), and consequently formula (IV), will be satisfied for arbitrary Δx_i, Δy_i, Δz_i, Δt, and in particular, therefore, also for all those which satisfy formula (III), or — which amounts to the same thing — formula (10). Thus the given motion satisfies Maupertuis' principle. However, it is obvious that for a suitable choice of forces, rest is impossible, i. e. rest is not an actual motion.

Remark 2. If the variation together with the variation of time were replaced in Maupertuis' principle by an ordinary variation, i. e. if we assumed that $\Delta t = 0$, writing δ for Δ, then formulae (III) and (IV) would assume the forms:

$$\delta' L = \delta E, \tag{14}$$

$$\delta \int_{t_0}^{t_1} E \, dt = 0, \text{ whence } \int_{t_0}^{t_1} \delta E \, dt = 0. \tag{15}$$

Therefore: for an actual motion formula (15) holds under the assumption that $\delta x_i, \delta y_i, \delta z_i$, are virtual displacements at every moment, equal to zero at $t = t_0$ and $t = t_1$, and satisfying condition (14).

The principle expressed in this manner, however, would not represent the characteristic properties of actual motions. For assuming e. g. that no forces act on a system, we should have $\delta' L = 0$. Consequently condition (14) would assume the form $\delta E = 0$ and hence it would imply formula (15) for every motion compatible with the constraints. Therefore in this case every motion compatible with the constraints, and not only an actual motion, would satisfy formula (15), i. e. Maupertuis' principle, in which the variation Δ is replaced by the ordinary variation δ.

We see from this that in Maupertuis' principle it is also essential that we form the variation together with the variation of time.

Remark 3. For a motion given in generalized coordinates it can be proved that *in holonomo-scleronomic systems Maupertuis' principle in the form* (IV) *holds under the assumption that the energy E is also expressed in generalized coordinates, and the variations Δq_j are virtual displacements equal to zero for $t = t_0$ and $t = t_1$ and satisfying condition* (III), *in which* $\Delta' L = \Sigma Q_j \, \Delta q_j$ (i. e. expressed in terms of the generalized forces Q_j).

Formula (IV) does not hold for rheonomic systems and generalized coordinates, and for them Maupertuis' principle is given in another form.