APPENDIX

ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH CONSTANT COEFFICIENTS

This is the name given to equations of the form

$$y'' + ay' + by = \varphi(x), \qquad (I)$$

where a, b, are given real numbers, $\varphi(x)$ is a known function; the sought for function satisfying (I) is $y = f(x)$.

Equation (I), in which the function $\varphi(x)$ is zero, is called a *homogeneous* equation.

A homogeneous equation therefore has the form

$$y'' + ay' + by = 0. \qquad (II)$$

In order to solve the homogeneous equation (II), we take

$$y = e^{rx}, \qquad (1)$$

where r is chosen so that equation (II) is satisfied.

Differentiating (1), we obtain:

$$y' = re^{rx}, \quad y'' = r^2e^{rx}. \qquad (2)$$

Substituting (1) and (2) in (II) we obtain

$$r^2e^{rx} + are^{rx} + be^{rx} = 0,$$

whence after dividing by e^{rx}

$$r^2 + ar + b = 0. \qquad (III)$$

Equation (III) is called the *characteristic equation* of (II).

The form of the solution of the homogeneous equation (II) depends on whether the roots r_1, r_2, of the characteristic equation (III) are real (equal or different), or complex. Let us therefore examine the three cases:

1° **Roots r_1, r_2, are real and different.** The most general solution of equation (II) is then

$$y = c_1e^{r_1x} + c_2e^{r_2x}, \qquad (3)$$

where c_1, c_2, are arbitrary constants.

2° Roots r_1, r_2, are real and equal. The most general solution of equation (II) is then

$$y = (c_1x + c_2)\,e^{r_1x}, \tag{4}$$

where c_1, c_2, are arbitrary constants.

3° Roots r_1, r_2, are complex. Since equation (III) has real coefficients a, b, then r_1, r_2, are conjugate imaginary numbers.

Let us take:

$$r_1 = \alpha + \beta i, \quad r_2 = \alpha - \beta i.$$

The most general solution of (II) is in this case

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x), \tag{5}$$

where c_1, c_2, are arbitrary constants.

In order to find the general solution of (I), we try first to find a particular solution of this equation. If we succeed and $y = \psi(x)$ is this particular solution, then we next solve the homogeneous equation (II). The most general solution of equation (I) is obtained by adding the particular solution $\psi(x)$ to the general solution of the homogeneous equation (II).

Example 1. Solve the equations:

$$\text{(a)}\ y'' - 3y' + 2y = 0; \quad \text{(b)}\ y'' + 2y' + y = 0;$$
$$\text{(c)}\ y'' - 2y' + 5y = 0.$$

The characteristic equations are:

$$\text{(a)}\ r^2 - 3r + 2 = 0; \quad \text{(b)}\ r^2 + 2r + 1 = 0;$$
$$\text{(c)}\ r^2 - 2r + 5 = 0.$$

The roots of these equations are:

$$\text{(a)}\ r_1 = 1,\ r_2 = 2; \quad \text{(b)}\ r_1 = r_2 = -1;$$
$$\text{(c)}\ r_1 = 1 + 2i,\ r_2 = 1 - 2i.$$

The most general solutions therefore have the form:

$$\text{(a)}\ y = c_1e^x + c_2e^{2x}; \quad \text{(b)}\ y = (c_1x + c_2)\,e^{-x};$$
$$\text{(c)}\ y = e^x(c_1 \cos 2x + c_2 \sin 2x).$$

Example 2. Solve the equation

$$\text{(d)}\ y'' - 3y' + 2y = 4x^2.$$

We try to find a solution of the form

$$y = ax^2 + bx + c. \tag{6}$$

In order to determine a, b, and c, we substitute (6) in (d). After forming derivatives, we get:

$$2a - 3(2ax + b) + 2(ax^2 + bx + c) = 4x^2,$$

whence

$$2ax^2 + (-6a + 2b)x + (2a - 3b + 2c) = 4x^2.$$

Equating coefficients, we obtain:

$$2a = 4, \quad -6a + 2b = 0, \quad 2a - 3b + 2c = 0;$$

consequently:

$$a = 2, \quad b = 6, \quad c = 7.$$

Therefore by (6) the particular solution of equation (d) is

$$y = 2x^2 + 6x + 7. \tag{7}$$

The homogeneous equation $y'' - 3y' + 2y = 0$ has the general solution

$$y = c_1e^x + c_2e^{2x} \tag{8}$$

(cf example 1 (a)). Therefore by (7) and (8) the most general solution of equation (d) is

$$y = c_1e^x + c_2e^{2x} + 2x^2 + 6x + 7,$$

where c_1, c_2, are arbitrary constants.