An example of an orthogonal development whose sum is everywhere different from the developed function*

The purpose of this note is to give an example of a Fourier-like development

$$f \sim c_1\psi_1 + c_2\psi_2 + \dots$$

of a summable function $f(t)$ defined in (a, b), $\{\psi_n(t)\}$ being a complete set of functions normalised and orthogonal in (a, b), such that

(i) the series $\sum_{n=1}^{\infty} c_n \psi_n(t)$ converges throughout (a, b), but

(ii) the sum of the series differs from $f(t)$ in every point t of (a, b).

The c_n are to be understood here as the Fourier constants of f, *i.e.*

$$c_n = \int_a^b f(t)\psi_n(t)\,dt \qquad (n = 1, 2, \dots), \tag{1}$$

and we shall choose our functions as to assure the existence of the integrals (1).

THEOREM. *Suppose that* (i) $f(t)$ *is defined throughout* (a, b), (ii) $f(t)$ *is positive, so that*

$$f(t) > 0 \qquad (a \leqslant t \leqslant b), \tag{2}$$

(iii) $f(t)$ *is summable in* (a, b), *and* (iv) $f^2(t)$ *is not summable. Then we can determine a complete, orthogonal, and normalised set of functions* $\{\psi_n(t)\}$, *defined and summable in* (a, b), *such that*

$$\int_a^b f(t)\psi_n(t)\,dt = 0 \qquad (n = 1, 2, \dots). \tag{3}$$

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* Commenté sur p. 318.

It is evident that our theorem implies the existence of the required example, because equations (3) and definitions (1) give immediately

$$\sum_{n=1}^{\infty} c_n \psi_n(t) \equiv 0 \qquad (a \leqslant t \leqslant b), \tag{4}$$

and from (2) and (4) it follows that

$$f(t) > \sum_{n=1}^{\infty} c_n \psi_n(t) \qquad (a \leqslant t \leqslant b). \tag{5}$$

Proof. Let $\{\varphi_n(t)\}$ be the ordinary complete and normalised *trigonometrical* set corresponding to (a, b) (1) and put

$$a_n = -\frac{\int_a^b f(t)\varphi_n(t)\,dt}{\int_a^b f(t)\,dt} \qquad (n = 1, 2, \ldots)\ (^2). \tag{6}$$

Then

$$\int_a^b [a_n + \varphi_n(t)] f(t)\,dt = 0 \qquad (n = 1, 2, \ldots). \tag{7}$$

The set $\{a_n + \varphi_n(t)\}$ *is complete*; in fact, let $\gamma(t)$ a function integrable together with its square in (a, b), and let us suppose

$$\int_a^b [a_n + \varphi_n(t)]\gamma(t)\,dt = 0 \qquad (n = 1, 2, \ldots), \tag{8}$$

$$\int_a^b \gamma^2(t)\,dt > 0. \tag{8'}$$

The so-called "Parseval-relation" holds for the trigonometrical set $\{\varphi_n(t)\}$ and gives

$$\sum_{n=1}^{\infty} \Big(\int_a^b \gamma(t)\varphi_n(t)\,dt\Big)^2 = \int_a^b \gamma^2(t)\,dt, \tag{9}$$

which, compared with (8) and (8'), implies

$$\sum_{n=1}^{\infty} \Big(a_n \int_a^b \gamma(t)\,dt\Big)^2 = \int_a^b \gamma^2(t)\,dt > 0. \tag{10}$$

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(1) $\varphi_1 = \dfrac{1}{\sqrt{(b-a)}}$, $\varphi_2 = \sqrt{\left(\dfrac{2}{b-a}\right)} \sin 2\pi \dfrac{t-a}{b-a}$, $\varphi_3 = \sqrt{\left(\dfrac{2}{b-a}\right)} \cos 2\pi \dfrac{t-a}{b-a}$, ...

(2) The denominator is positive, by (2).

It follows that $\int_a^b \gamma(t)\,dt \neq 0$, and so that the series

$$\sum_{n=1}^{\infty} a_n^2 \tag{11}$$

is convergent. But this implies, by (6), the convergence of

$$\sum_{n=1}^{\infty} \Big(\int_a^b f(t)\,\varphi_n(t)\,dt\Big)^2, \tag{12}$$

and so the existence of $\int_a^b f^2(t)\,dt$, which must be equal to (12). This is contradictory to our hypotheses; and this contradiction shows that the assumptions (8) and (8′) are incompatible, and so that the set $\{a_n+\varphi_n(t)\}$ is complete.

Put

$$a_n+\varphi_n(t) = \chi_n(t) \qquad (n = 1, 2, \ldots). \tag{13}$$

Then $\{\chi_n(t)\}$ is a *complete set of continuous functions*, and we have, by (7) and (13),

$$\int_a^b f(t)\,\chi_n(t)\,dt = 0 \qquad (n = 1, 2, \ldots). \tag{14}$$

We have now only to derive from $\{\chi_n(t)\}$ a new set $\{\psi_n(t)\}$ by the "orthogonalisation method" of Mr. E. Schmidt to get a complete, orthogonal and normalised set possessing the property (3). The ψ_n are linear finite forms in χ_n, and thus our set is composed of trigonometrical polynomials.
