PREFACE.

This edition differs from the first [1]) by the new arrangement of the contents of several chapters, some of which have been completed by more recent results, and by the suppression of a number of errors, obligingly pointed out by Mr. V. Jarnìk, which formed the object of the two pages of Errata in the first edition. It is probable that fresh errors have slipped in owing to modifications of the text, but the reader would certainly find many more, if the author had not received the valuable help of Messrs J. Todd, A. J. Ward and A. Zygmund in reading the proofs. Also, Mr. L. C. Young has greatly exceeded his rôle of translator in his collaboration with the author. To all these I express my warmest thanks.

This volume contains two Notes by S. Banach. The first of them, on Haar's measure, is the translation (with a few slight modifications) of the note already contained in the French edition of this book. The second, which concerns the integration in abstract spaces, is published here for the first time and completes the considerations of Chapter I.

The numbers given in the bibliographical references relate to the list of cited works which will be found at the end of the book. The asterisks preceding certain titles indicate the parts of the book which may be omitted on first reading.

S. Saks.

Warszawa-Żoliborz, July, 1937.

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1) S. Saks, *Théorie de l'Intégrale,* Monografie Matematyczne, Volume II, Warszawa 1933.

FROM THE PREFACE TO THE FIRST EDITION.

The modern theory of real functions became distinct from classical analysis in the second half of the 19-th century, as a result of researches, unsystematic at first, which dealt with the foundations of the Differential Calculus or which concerned the discovery of functions whose properties appeared to be very strange and unexpected.

The distrust with which this new field of investigation was regarded is typified by the attitude of H. Poincaré who wrote:" *Autrefois quand on inventait une fonction nouvelle, c'était en vue de quelque but pratique; aujourd'hui on les invente tout exprès pour mettre en défaut les raisonnements de nos pères et on n'en tirera jamais que celà*".

This view was by no means isolated. Ch. Hermite, in a letter to T. J. Stieltjes, expressed himself in even stronger terms: "*Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions qui n'ont pas de dérivées*". Researches dealing with non-analytic functions and with functions violating laws which one hoped were universal, were regarded almost as the propagation of anarchy and chaos where past generations had sought order and harmony. Even the first attempts to establish a positive theory were rather sceptically received: it was feared that an excessively pedantic exactitude in formulating hypotheses would spoil the elegance of classical methods, and that discussions of details would end by obscuring the main ideas of analysis. It is true that the first researches hardly went beyond the traditional, formal apparatus, fixed by Cauchy and Riemann, which was difficult to adapt to the requirements of the new problems. Nevertheless, these researches succeeded in opening the way to applications of the Theory of Sets to Analysis, and — to quote H. Lebesgue's inaugural lecture at the Collège de France — "*the great authority of Camille Jordan gave to the new school a valuable encouragement which amply compensated the few reproofs it had to suffer*".

R. Baire, E. Borel, H. Lebesgue — these are the names which represent the Theory of Real Functions, not merely as an object of researches, but also as a method, names which at the same time recall the leading ideas of the theory. The names of Baire and Borel will be always associated with the method of classification of functions and sets in a transfinite hierarchy by means of certain simple operations to which they are subjected. Excellent accounts of this subject are to be found in the treatises: Ch. J. de la Vallée Poussin, *Fonctions d'ensemble, Intégrale de Lebesgue, Classes de Baire*, 1916, F. Hausdorff, *Mengenlehre*, 1927, H. Hahn, *Theorie der reellen Funktionen*, 1933 (recent edition), C. Kuratowski, third volume of the present collection, and finally in the book of W. Sierpiński, *Topologja ogólna* (in Polish), and its English translation, *General Topology*, to be published in 1934 by the Toronto University Press.

The other line of researches, which arises directly from the study of the foundations of the Integral Calculus, is still more intimately connected with the great trains of thought of Analysis in the last century. On several occasions attempts were made to generalize the old process of integration of Cauchy-Riemann, but it was Lebesgue who first made real progress in this matter. At the same time, Lebesgue's merit is not only to have created a new and more general notion of integral, nor even to have established its intimate connection with the theory of measure: the value of his work consists primarily in his theory of derivation which is parallel to that of integration. This enabled his discovery to find many applications in the most widely different branches of Analysis and, from the point of view of method, made it possible to reunite the two fundamental conceptions of integral, namely that of definite integral and that of primitive, which appeared to be forever separated as soon as integration went outside the domain of continuous functions.

The theory of Lebesgue constitutes the subject of the present volume. While distinguishing it from that of Baire, we have no wish to erect an artificial barrier between two streams of thought which naturally intermingle. On the contrary, we shall have frequent occasion, particularly in the last chapters of this book, to show explicitly how Lebesgue's theory comes to be bound up not only with the results, but also with the very methods, of the theory of Baire. Is not the idea of Denjoy integration at bottom merely a striking adaptation of the idea which guided Baire? Where Baire, by repeated application of passage to the limit, widened the class of functions, Denjoy constructed a transfinite hierarchy of methods of integration starting with that of Lebesgue and whose successive stages are connected by two operations: one corresponding exactly to the generalized integral of Cauchy and the other to the generalized integral of Harnack-Jordan.

Now that the Theory of Real Functions, while losing perhaps a little of the charm of its first youth, has ceased to be a "new" science, it seems superfluous to discuss its importance. It is known that the theory has brought to light regularity and harmony, unhoped for by the older methods, concerning, for instance, the existence of a limit, a derivative, or a tangent. It is enough to mention the theorems, now classical, on the behaviour of a power series on, or near, the boundary of its circle of convergence. Also, many branches of analysis, to cite only Harmonic Analysis, Integral Equations, Functional Operations, have lost none of their elegance where they have been inspired by methods of the Theory of Real Functions. On the contrary, we have learnt to admire in the arguments not only cleverness of calculation, but also the generality which, by an apparent abstraction, often enables us to grasp the real nature of the problem.

The object of the preceding remarks has been to indicate the place occupied by the subject of this volume in the Theory of Real Functions[1]). Let us now say a few words about the structure of the book. It embodies the greater part of a course of lectures delivered by the author at the University of Warsaw (and published in Polish in a separate book [2])), which has been modified and completed by several chapters. The reader need only be acquainted with a few elementary principles of the Theory of Sets, which are to be found in most courses of lectures on elementary analysis. Actually a summary of the elements of the theory of sets of points is given in one of the opening paragraphs.

Several pages of the book are inspired by suggestions and methods which I owe to the excellent university lectures of my teacher, W. Sierpiński, the influence of whose ideas has often guided my personal researches. Finally, I wish to express my warmest thanks to all those who have kindly assisted me in my task, particularly to my friend A. Zygmund, who undertook to read the manuscript. I thank also Messrs C. Kuratowski and H. Steinhaus for their kind remarks and bibliographical indications.

S. Saks.

Warszawa, May, 1933.

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[1]) In this preface, I made no attempt to write a history of the early days of the theory, and still less to settle questions of priority of discovery. But, since an English Edition of this book is appearing now, I think I ought to mention the name of W. H. Young, whose work on the theory of integration started at the same period as that of Lebesgue.

[2]) *Zarys teorji całki,* Warszawa 1930, Wydawnictwo Kasy im. Mianowskiego, Instytutu Popierania Nauki.

MONOGRAFIE MATEMATYCZNE
KOMITET REDAKCYJNY:

TOM VII

THEORY OF THE INTEGRAL

BY

DR. STANISŁAW SAKS

(SECOND REVISED EDITION)

ENGLISH TRANSLATION
BY
L. C. YOUNG, M. A.

WITH TWO ADDITIONAL NOTES
BY
PROF. DR. STEFAN BANACH

Z SUBWENCJI FUNDUSZU KULTURY NARODOWEJ

WARSZAWA — LWÓW 1937

NEW YORK: G. E. STECHERT & Co.
31 EAST 10 th STREET

PRINTED IN POLAND

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO
POD ZARZĄDEM JÓZEFA FILIPOWSKIEGO
PRINTED IN POLAND