NOTE II.

The Lebesgue integral in abstract spaces

by

Stefan Banach.

Introduction.

In this note we intend to establish some general theorems concerning the Lebesgue integral in abstract spaces. This subject has been discussed by several authors (for the references see this volume, pp. 4, 88, 116, 156 and 157). Our considerations differ from those of other writers in that they are not based on the notion of measure.

Let us fix a set of arbitrary elements $\boldsymbol{H}$ as an abstract space. We shall denote real functions (i. e. functions which admit real values) defined in $\boldsymbol{H}$ by $x(t)$, $y(t)$, $z(t)$,... where $t \in \boldsymbol{H}$, or simply by $x, y, z, \ldots$. A set $\mathfrak{L}$ of real functions defined in $\boldsymbol{H}$ will be called *linear* if any linear combination, with constant coefficients, of two elements of $\mathfrak{L}$, also belongs to $\mathfrak{L}$.

Let $\mathfrak{L}$ be a linear set of functions defined in $\boldsymbol{H}$. A functional F defined in $\mathfrak{L}$ is termed *additive* if for any pair of elements x and y of $\mathfrak{L}$ and any real number α, we have $F(x+y)=F(x)+F(y)$ and $F(\alpha x)=\alpha \cdot F(x)$. The functional F is *non-negative* if $F(x) \geqslant 0$ for any non-negative function $x \in \mathfrak{L}$.

We say that a functional F defined in $\mathfrak{L}$ is a *Lebesgue integral* (*$\mathfrak{L}$-integral*) in $\mathfrak{L}$ if the following conditions are satisfied:

A) The set $\mathfrak{L}$ is linear;

B) the functional F is additive and non-negative;

C) if 1^0 $\{z_n\} \subset \mathfrak{L}$ and $M \in \mathfrak{L}$, 2^0 $|z_n(t)| \leqslant M(t)$ for $n=1,2,\ldots$ and $t \in \boldsymbol{H}$, and 3^0 $\lim_n z_n(t)=z(t)$ for $t \in \boldsymbol{H}$, then $z \in \boldsymbol{H}$ and $\lim_n F(z_n)=F(z)$;

D) if $z \in \mathfrak{L}$, $F(z)=0$ and $|y(t)| \leqslant z(t)$ for $t \in \boldsymbol{H}$, then $y \in \mathfrak{L}$ and $F(y)=0$;

E) if 1^0 $\{z_n\} \subset \mathfrak{L}$, $z_n(t) \leqslant z_{n+1}(t)$ for $n=1,2,\ldots$, 2^0 $\lim_n z_n(t)=z(t)$ for $t \in \boldsymbol{H}$, and 3^0 $\lim_n F(z_n) < +\infty$, then $z \in \mathfrak{L}$ and $\lim_n F(z_n)=F(z)$.

The Lebesgue integrals considered in this note will moreover satisfy the condition:

R) If $z \in \mathfrak{L}$, then $|z| \in \mathfrak{L}$.

In Part I, a condition is established under which an additive and non-negative functional defined in a linear set of functions $\mathfrak{E}$, may be extended to an $\mathfrak{L}$-integral on a certain set $\mathfrak{L}$ containing $\mathfrak{E}$. The $\mathfrak{L}$-integral and the set $\mathfrak{L}$ will be explicitly defined.

In Part II we admit that $\boldsymbol{H}$ is a metrical and compact space. We consider an $\mathfrak{L}$-integral defined in sets containing all functions which are bounded and measurable in the Borel sense. It is shown that each $\mathfrak{L}$-integral of this kind is determined by the values which it admits for continuous functions. Conversely, any additive and non-negative functional defined for all continuous functions may be extended as an $\mathfrak{L}$-integral to the class of functions measurable $(\mathfrak{B})$. We thus obtain the most general $\mathfrak{L}$-integral defined for all functions bounded and measurable $(\mathfrak{B})$.

In Part III we deal with an analogous problem supposing that $\boldsymbol{H}$ is the unit sphere of the Hilbert space. In particular, the integral of a continuous function is expressed by explicit formulae.

I. Abstract sets.

§ 1. We shall employ the following notations:

1. $x \geqslant y$ if $x(t) \geqslant y(t)$ for every $t \in \boldsymbol{H}$; in particular $x \geqslant 0$ means that $x(t) \geqslant 0$ for $t \in \boldsymbol{H}$;

2. $|x|=|x(t)|$ is the modulus of $x(t)$ in the ordinary sense;

3. $\max(x,y)=\frac{1}{2}(x+y+|x-y|)$, $\min(x,y)=\frac{1}{2}(x+y-|x-y|)$;

4. $\lim_n x_n=x$ means that $\lim_n x_n(t)=x(t)$ for $t \in \boldsymbol{H}$; the relations $\limsup_n x_n=x$, $\liminf_n x_n=x$ are defined similarly;

5. $\overset{+}{x}=\frac{1}{2}(x+|x|)$, $\overset{-}{x}=\frac{1}{2}(x-|x|)$ (cf. Chap. I, p. 13).

§ 2. For the rest of Part I of this note we shall fix a set $\mathfrak{E}$ of real functions defined in $\boldsymbol{H}$, and a functional $f(x)$ defined for $x \in \mathfrak{E}$, subject to the following conditions:

(i_1) The set $\mathfrak{E}$ is linear;

(i_2) if $x \in \mathfrak{E}$, then $|x| \in \mathfrak{E}$;

(ii_1) the functional f is additive;

(ii_2) the functional f is non-negative;

(ii_3) if 1^0 $\{x_n\} \subset \mathfrak{E}$ and $M \in \mathfrak{E}$, 2^0 $|x_n| \leqslant M$ for $n=1,2,\ldots$, and 3^0 $\lim\limits_n x_n = 0$, then $\lim\limits_n f(x_n) = 0$.

It follows immediately from the conditions (i) that for any pair of elements x and y of $\mathfrak{E}$, $\max(x,y)$, $\min(x,y)$, $\mathring{x}$ and $\mathring{\mathring{x}}$ also belong to $\mathfrak{E}$. It follows further that the condition (ii_3) is equivalent to the following condition:

(ii_3') If 1^0 $\{x_n\} \subset \mathfrak{E}$ and $m \in \mathfrak{E}$, 2^0 $x_n \geqslant m$ for $n=1,2,\ldots$, and 3^0 $\liminf\limits_n x_n \geqslant 0$, then $\liminf\limits_n f(x_n) \geqslant 0$.

§ 3. We shall establish the following

Theorem 1. *If the set $\mathfrak{E}$ and the functional f satisfy the conditions* (i) *and* (ii), *then there exists an $\mathfrak{L}$-integral F, defined in a set $\mathfrak{L}$ containing $\mathfrak{E}$, such that $F(x)=f(x)$ whenever $x \in \mathfrak{E}$; moreover, this integral satisfies the condition* R).

The proof will result from several lemmas.

§ 4. We denote by $\mathfrak{L}^*$ the set of all functionals $z(t)$ defined in $\boldsymbol{H}$ for each of which there exist two sequences $\{x_n\} \subset \mathfrak{E}$, $\{y_n\} \subset \mathfrak{E}$ such that

$$\liminf_n x_n \geqslant z \geqslant \limsup_n y_n. \tag{1}$$

It is easily seen that the set $\mathfrak{L}^*$ is linear and that $\mathfrak{E} \subset \mathfrak{L}^*$.

Given a function $z \in \mathfrak{L}^*$, we shall term *upper $\mathfrak{L}$-integral* of z the lower bound of all (finite or infinite) numbers g for each of which there exist a function $m \in \mathfrak{E}$ and a sequence of functions $\{x_n\}$ belonging to $\mathfrak{E}$ such that $x_n \geqslant m$ for $n=1,2,\ldots$, $\liminf\limits_n x_n \geqslant z$ and $g = \liminf\limits_n f(x_n)$.

The definition of the *lower $\mathfrak{L}$-integral* is analogous to that of the upper $\mathfrak{L}$-integral. The upper and lower $\mathfrak{L}$-integrals of a function $z \in \mathfrak{L}^*$ will be denoted by $p(z)$ and $q(z)$ respectively. We obviously have $q(z) = -p(-z)$.

§ 5. The sequence $\{f(x_n)\}$ in the above definition of the upper $\mathfrak{L}$-integral, may obviously be supposed convergent (to a finite limit or $+\infty$). Further, if $\{x_n\}\subset\mathfrak{E}$, $m\in\mathfrak{E}$, $z\geqslant 0$, $x_n\geqslant m$ for $n=1,2,\ldots$ and $\liminf_n x_n\geqslant z$, then $\lim_n x_n=0$ and consequently, by the condition (ii$_3$), § 2, $\lim_n f(x_n)=0$. Hence, *if* $z\in\mathfrak{L}^*$, $z\geqslant 0$ *and* $p(z)<P<+\infty$, *there always exists a sequence of non-negative functions* $\{x_n\}$ *belonging to* $\mathfrak{E}$ *such that* $\liminf_n x_n\geqslant z$ *and* $f(x_n)<P$ *for* $n=1,2,\ldots$

Lemma 1. *For any function* $x\in\mathfrak{E}$ *we have* $p(x)=f(x)$.

Proof. Writing $x_n=x$ and $m=x$, we have

$$\liminf_n x_n\geqslant x \quad \text{and} \quad x_n\geqslant m \quad \text{for} \quad n=1,2,\ldots, \tag{1}$$

whence $p(x)\leqslant f(x)$. On the other hand, if $x_1, x_2, \ldots$ and m are any functions which belong to $\mathfrak{E}$ and satisfy the relations (1), then $\liminf_n (x_n-x)\geqslant 0$ and $x_n-x\geqslant m-x$ for $n=1,2,\ldots$. It follows from (ii$_3'$), § 2, that $\liminf_n f(x_n-x)\geqslant 0$, i. e. $\liminf_n f(x_n)\geqslant f(x)$. Thus $p(x)\geqslant f(x)$, and finally $p(x)=f(x)$.

Lemma 2. *If* $z_1\in\mathfrak{L}^*$, $z_2\in\mathfrak{L}^*$ *and if, moreover,* $p(z_1)<+\infty$, $p(z_2)<+\infty$, *then* $p(z_1+z_2)\leqslant p(z_1)+p(z_2)$.

Proof. Let P_1 and P_2 be arbitrary numbers such that $p(z_1)<P_1$ and $p(z_2)<P_2$. There exist two sequences $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$ of functions belonging to $\mathfrak{E}$ and two functions $m_1\in\mathfrak{E}$ and $m_2\in\mathfrak{E}$ such that $\liminf_n x_n^{(j)}\geqslant z_j$ and $\lim_n f(x_n^{(j)})<P_j$ for $j=1,2$ and such that $x_n^{(j)}>m_j$ for $j=1,2$ and $n=1,2,\ldots$. Therefore, writing $x_n=x_n^{(1)}+x_n^{(2)}$ and $m=m_1+m_2$, we have $\liminf_n x_n\geqslant z_1+z_2$ and $x_n\geqslant m$ for $n=1,2,\ldots$. Consequently $p(z_1+z_2)\leqslant\lim_n f(x_n)=\lim_n f(x_n^{(1)})+\lim_n f(x_n^{(2)})<P_1+P_2$, whence $p(z_1+z_2)\leqslant p(z_1)+p(z_2)$.

Lemma 3. *For any function* $z\in\mathfrak{L}^*$, *we have* $p(z)\geqslant q(z)$.

Proof. Since $q(z)=-p(-z)$ (cf. § 4), the inequality $p(z)\geqslant q(z)$ is obvious if one of the numbers $p(z)$ or $p(-z)$ is $+\infty$; while, if $p(z)<+\infty$ and $p(-z)<+\infty$, it follows immediately from Lemma 2.

Lemma 4. *If* $z \in \mathfrak{L}^*$, $p(z) < +\infty$, *then also* $p(\mathring{z}) < +\infty$ *and* $p(z) = p(\mathring{z}) + p(\mathring{z})$.

Proof. Given an arbitrary finite number $P > p(z)$, there exist a function $m \in \mathfrak{E}$ and a sequence $\{x_n\}$ of functions belonging to $\mathfrak{E}$ such that $x_n \geqslant m$ for $n = 1, 2, \ldots$, $\liminf_n x_n \geqslant z$ and $\limsup_n f(x_n) < P$. Note that $\mathring{x}_n \geqslant \mathring{m}$, and consequently $f(\mathring{x}_n) \leqslant f(x_n) - f(\mathring{m})$, for $n = 1, 2, \ldots$, whence $p(\mathring{z}) \leqslant \liminf_n f(\mathring{x}_n) < +\infty$. Again

$$P > \limsup_n f(x_n) \geqslant \liminf_n f(\mathring{x}_n) + \liminf_n f(\mathring{x}_n) \geqslant p(\mathring{z}) + p(\mathring{z}),$$

and therefore $p(z) \geqslant p(\mathring{z}) + p(\mathring{z})$; whence, in virtue of Lemma 2, $p(z) = p(\mathring{z}) + p(\mathring{z})$.

Finally, we mention two propositions which are directly obvious:

Lemma 5. *If* $z_1 \in \mathfrak{L}^*$, $z_2 \in \mathfrak{L}^*$ *and* $z_1 \leqslant z_2$, *then* $p(z_1) \leqslant p(z_2)$; *in particular, if* $z \in \mathfrak{L}^*$ *and* $z \geqslant 0$, *then* $p(z) \geqslant 0$.

Lemma 6. *If* $z \in \mathfrak{L}^*$, *then* $p(\lambda z) = \lambda \cdot p(z)$ *for any non-negative number* λ.

§ 6. We shall now denote by $\mathfrak{L}$ the set of all functions $z \in \mathfrak{L}^*$ for which $p(z) = q(z) \neq \infty$. The following proposition is an immediate consequence of Lemmas 2 and 6:

Lemma 7. *If* $z_1 \in \mathfrak{L}$ *and* $z_2 \in \mathfrak{L}$, *then* $(\lambda_1 z_1 + \lambda_2 z_2) \in \mathfrak{L}$ *and* $p(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 p(z_1) + \lambda_2 p(z_2)$ *for any pair of finite numbers* λ_1 *and* λ_2.

Lemma 8. *If* $z \in \mathfrak{L}$, *then* $|z| \in \mathfrak{L}$.

Proof. Since $|z| = \breve{z} - \breve{z}$, it is enough to prove that $\breve{z} \in \mathfrak{L}$ and $\breve{z} \in \mathfrak{L}$. To this end, let us remark that, in virtue of Lemma 4, $p(\mathring{z}) < +\infty$, $p(\mathring{z}) > -\infty$ and $p(z) = p(\mathring{z}) + p(\mathring{z})$; by symmetry, $q(\mathring{z}) > -\infty$, $q(\mathring{z}) < +\infty$ and $q(z) = q(\mathring{z}) + q(\mathring{z})$. Since, by hypothesis, $p(z) = q(z)$, it follows that $[p(\mathring{z}) - q(\mathring{z})] + [p(\mathring{z}) - q(\mathring{z})] = 0$, and so by Lemma 3, $p(\mathring{z}) = q(\mathring{z}) \neq \infty$ and $p(\mathring{z}) = q(\mathring{z}) \neq \infty$.

Lemma 9. *If* z *is the limit of a non-decreasing sequence* $\{z_n\}$ *of functions belonging to* $\mathfrak{L}$ *and* $\limsup_n p(z_n) < +\infty$, *then* $z \in \mathfrak{L}$ *and* $p(z) = \limsup_n p(z_n)$.

Proof. We can clearly assume (by subtracting, if necessary, the function z_1 from all functions of the sequence $\{z_n\}$) that $z_1 = 0$. Writing $w_n = z_{n+1} - z_n$ for $n = 1, 2, \ldots$, we shall now follow an argument similar to that of Theorem 12.3, Chap. I. First, we have $z \geqslant z_n$ and $p(z_n) = q(z_n)$ for every n, and so

$$q(z) \geqslant \limsup_n q(z_n) = \limsup_n p(z_n). \tag{1}$$

To establish the opposite inequality, let ε be an arbitrary positive integer and let us associate (cf. the remark at the beginning of § 5) with each function w_n a sequence $\{x_n^{(k)}\}_{k=1,2,\ldots}$ of non-negative functions belonging to $\mathfrak{E}$ such that

$$(2)\qquad \liminf_k x_n^{(k)} \geqslant w_n \qquad \text{and} \qquad (3)\qquad f(x_n^{(k)}) \leqslant p(w_n) + \varepsilon/2^n.$$

Let us write $y_k = \sum_{n=1}^{k} x_n^{(k)}$. The functions y_k clearly belong to $\mathfrak{E}$ and, by (2), we have $\liminf_k y_k \geqslant \sum_k w_k = z$. On the other hand, in virtue of (3), we find $f(y_k) \leqslant \sum_{n=1}^{k} p(w_n) + \varepsilon \leqslant p(z_{k+1}) + \varepsilon \leqslant \lim_i p(z_i) + \varepsilon$ for $k = 1, 2, \ldots$. Therefore, $p(z) \leqslant \liminf_k f(y_k) \leqslant \lim_k p(z_k) + \varepsilon$, and since ε is an arbitrary positive number, this combined with (1) gives $0 \leqslant p(z) = q(z) = \lim_k p(z_k) < +\infty$, which completes the proof.

Lemma 10. *If $M \in \mathfrak{L}$ and $\{z_n\}$ is a sequence of functions belonging to $\mathfrak{L}$ such that $|z_n| \leqslant M$ for $n = 1, 2, \ldots$, then, putting $g = \liminf_n z_n$ and $h = \limsup_n z_n$, we have $g \in \mathfrak{L}$, $h \in \mathfrak{L}$, and*

$$p(g) \leqslant \liminf_n p(z_n) \leqslant \limsup_n p(z_n) \leqslant p(h).$$

Consequently, if the sequence $\{z_n\}$ is convergent and $z = \lim_n z_n$, then $p(z) = \lim_n p(z_n)$.

Proof. The lemma corresponds to Theorem 12.11, Chap. I, and its proof is analogous to that of the latter. Let us write, for each pair of integers i and $j \geqslant i$, $g_{ij} = \min(z_i, z_{i+1}, \ldots, z_j)$. The sequence $\{g_{ij}\}_{j=i,\,i+1,\ldots}$ is non-increasing, and consequently the sequence $\{M - g_{ij}\}_{j=i,\,i+1,\ldots}$ is non-decreasing. Let $g_i = \lim_j g_{ij}$. Since the functions g_{ij} clearly belong to $\mathfrak{L}$, it follows from Lemma 9 that $M - g_i \in \mathfrak{L}$ and $p(M - g_i) = \lim_j p(M - g_{ij})$, i. e. $g_i \in \mathfrak{L}$ and $p(g_i) = \lim_j p(g_{ij})$. Hence, applying again Lemma 9 to the non-decreasing sequence $\{g_i\}$ which converges to g, we obtain $g \in \mathfrak{L}$ and

$$p(g) = \lim_i p(g_i) \leqslant \liminf_i p(z_i).$$

By symmetry we have the analogous result for h and the proof is complete.

We shall conclude this § by mentioning the following lemma which is an immediate consequence of Lemma 5:

Lemma 11. *If $z \in \mathfrak{L}$, $z \geqslant 0$ and $p(z) = 0$, then any function x such that $|x| \leqslant z$ belongs to $\mathfrak{L}$ and for any such function x we have $p(x) = 0$.*

§ 7. Let $F(x)=p(x)$ for $x \in \mathfrak{L}$. The lemmas of the preceding sections show that the set $\mathfrak{L}$ and the functional $F(x)$ satisfy the theorem stated in § 3. Theorem 1 is thus proved.

It is easily seen that if an $\mathfrak{L}$-integral F_1 defined in a linear set $\mathfrak{L}_1 \supset \mathfrak{L}$ satisfies the condition $f(x)=F_1(x)$ for $x \in \mathfrak{E}$, then $F(x)=F_1(x)$ for all $x \in \mathfrak{L}$. Consequently the functional f determines completely an $\mathfrak{L}$-integral in the set $\mathfrak{L}$.

II. Metrical compact sets.

§ 8. Let now $\boldsymbol{H}$ be a complete and compact metrical space. We shall specify $\mathfrak{E}$ as the set of functions continuous in $\boldsymbol{H}$.

The set $\mathfrak{E}$ satisfies evidently the conditions (i), § 2. It may be shown that any additive and non-negative functional f defined in $\mathfrak{E}$ satisfies the condition (ii_3) [1]).

Theorem 1 permits to define a Lebesgue integral $F(x)$ for all functions x belonging to a certain set $\mathfrak{L} \supset \mathfrak{E}$, in such a manner that the condition R), p. 321, is satisfied and that $F(x)=f(x)$ for $x \in \mathfrak{E}$.

Evidently, every function $x(t)$ which is constant on $\boldsymbol{H}$ belongs to $\mathfrak{E}$. It follows by condition C), p. 320, that every bounded function measurable in the sense of Borel belongs to $\mathfrak{E}$.

We have thus proved the following

Theorem 2. *Every additive and non-negative functional, defined for all functions which are continuous in a complete compact space* $\boldsymbol{H}$, *may by extended to an* $\mathfrak{L}$*-integral defined in a certain linear set (containing all bounded functions measurable in the sense of Borel) so that the condition* R) *be satisfied.*

The values of this $\mathfrak{L}$-integral for functions bounded and measurable $(\mathfrak{B})$ are, of course, determined by the given functional f. Hence the most general $\mathfrak{L}$-integral defined for this class of functions may be obtained by choosing an arbitrary additive non-negative functional defined for all functions which are continuous in $\boldsymbol{H}$ and by extending this functional by means of the method described in Part I of this note.

---

[1]) A functional of this kind is necessarily linear. Every linear functional defined in $\mathfrak{E}$ satisfies the condition (ii_3). See S. Banach [I, p.224].

Any linear functional $f(x)$ defined in the set E is the difference of two additive non-negative functionals $f_1(x)$ and $f_2(x)$ (cf. S. Banach [I, p. 217]). Extending these functionals by means of Theorem 1 over two sets, $\mathfrak{L}_1$ and $\mathfrak{L}_2$ say, respectively, we see that it is possible to extend the functional $f(x)$ over the linear set $\mathfrak{L}=\mathfrak{L}_1\cdot\mathfrak{L}_2$. This set will contain all bounded functions measurable $(\mathfrak{B})$. The extended additive functional $F(x)$ evidently satisfies the conditions C) and R), p. 321, and is non-negative.

III. The Hilbert space.

§ 9. We shall now understand by $\boldsymbol{H}$ the unit sphere of the Hilbert space, i. e. the set of all sequences $\{\vartheta_i\}$ for which $\sum_{i=1}^{\infty}\vartheta_i^2\leqslant 1$. The distance of two points $t=\{\vartheta_i\}$ and $t'=\{\vartheta_i'\}$ is defined, as usually, by the formula

$$\varrho(t,t')=[\sum_{i=1}^{\infty}(\vartheta_i-\vartheta_i')^2]^{\frac{1}{2}}.$$

With regard to this definition of distance the space $\boldsymbol{H}$ is not compact and therefore we cannot apply Theorem 2 directly.

Let $\mathfrak{E}_n$ be the set of functions $x=x(t)=x(\vartheta_1,\vartheta_2,\ldots)$ which are continuous in $\boldsymbol{H}$ and whose values depend only on the first n coordinates ϑ_j, so that $x(\vartheta_1,\vartheta_2,\ldots)=x(\vartheta_1,\vartheta_2,\ldots,\vartheta_n,0,0,\ldots)$ for any $t=\{\vartheta_i\}\in\boldsymbol{H}$. Clearly $\mathfrak{E}_n\subset\mathfrak{E}_{n+1}$.

It is easily seen that the set $\mathfrak{E}=\sum_{n=1}^{\infty}\mathfrak{E}_n$ satisfies the conditions (i), § 2. Any functional f defined in $\mathfrak{E}$ for which the conditions (ii) hold may be extended to an $\mathfrak{L}$-integral defined in a certain set $\mathfrak{L}$ containing $\mathfrak{E}$.

Lemma 12. *The set $\mathfrak{L}$ contains all bounded functions mesurable $(\mathfrak{B})$ defined in $\boldsymbol{H}$.*

Proof. Let x be a bounded continuous function defined in $\boldsymbol{H}$. For any point $t=(\vartheta_1,\vartheta_2,\ldots,\vartheta_n,\ldots)$ and any positive integer n, we write $x_n(t)=x(\vartheta_1,\ldots,\vartheta_n,0,0,\ldots)$. Evidently $x_n\in\mathfrak{E}$ and $\lim_n x_n=x$. If M is the upper bound of $|x(t)|$ for $t\in\boldsymbol{H}$, then $|x_n|\leqslant M$. Since the constant function $z=M$ certainly belongs to $\mathfrak{E}$, it follows from the condition C), p. 320, that $x\in\mathfrak{L}$.

Consequently every bounded and continuous function belongs to $\mathfrak{L}$ and by the condition C) the same is true of any bounded function measurable ($\mathfrak{B}$).

Lemma 13. *Every additive and non-negative functional $f(x)$ defined in $\mathfrak{E}$ satisfies the condition* (ii_3), § 2.

Proof. We define in $\boldsymbol{H}$ a distance $\varrho_1(t, t')$ of two points $t=\{\vartheta_1, \vartheta_2, \ldots\}$, $t'=\{\vartheta_1', \vartheta_2', \ldots\}$ by

$$\varrho_1(t, t') = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\vartheta_i - \vartheta_i'|}{1 + |\vartheta_i - \vartheta_i'|}. \tag{1}$$

We easily verify that with regard to this distance the set $\boldsymbol{H}$ is complete and compact.

Let $\widetilde{\mathfrak{E}}$ be the set of all functions defined in $\boldsymbol{H}$ which are continuous according to the distance defined by the formula (1). Evidently $\mathfrak{E} \subset \widetilde{\mathfrak{E}}$.

Let f be an additive non-negative functional defined in $\mathfrak{E}$. Let $x_n(t)=x(\vartheta_1, \ldots, \vartheta_n, 0, 0, \ldots)$ for $x \in \widetilde{\mathfrak{E}}$ and $t=(\vartheta_1, \vartheta_2, \ldots) \in \boldsymbol{H}$.

With regard to the distance (1), $\boldsymbol{H}$ is a complete and compact space, and hence the function $x(t) \in \mathfrak{E}$ is uniformly continuous. It follows that the sequence $\{x_n\}$ uniformly converges to x. This implies the convergence of the sequence $\{f(x_n)\}$ [1]. Let $\widetilde{f}(x) = \lim_n f(x_n)$.

If $x \geqslant 0$, then $x_n \geqslant 0$ for each n, and consequently $\widetilde{f}(x) \geqslant 0$. The functional $\widetilde{f}(x)$, clearly additive, is therefore non-negative. The set $\boldsymbol{H}$ being compact, it follows, by what has been established in Part II, that $\widetilde{f}$ satisfies in $\boldsymbol{H}$ the condition (ii_3) (with $\mathfrak{E}$ and f replaced by $\widetilde{\mathfrak{E}}$ and $\widetilde{f}$ respectively). Since $\mathfrak{E} \subset \widetilde{\mathfrak{E}}$ and $\widetilde{f}(x)=f(x)$ for $x \in \mathfrak{E}$, the functional f satisfies the condition (ii_3) in $\mathfrak{E}$.

§ 10. Now consider an additive non-negative functional $f(x)$ defined in $\mathfrak{E}$. Let $f_n(x)$ denote the functional defined in $\mathfrak{E}_n$ by the formula

$$f_n(x) = f(x) \qquad \text{for} \quad x \in \mathfrak{E}_n. \tag{2}$$

We obviously have

$$f_n(x) = f_{n+1}(x) \qquad \text{for} \quad x \in \mathfrak{E}_n. \tag{3}$$

[1] Indeed if $\varepsilon > 0$, there exists a positive integer N such that $-\varepsilon \leqslant x_p - x_q \leqslant \varepsilon$ whenever $p > N$, $q > N$. Since the constant function $z=1$ belongs to $\mathfrak{E}$, we have, for $k=f(1)$, the inequality $-k\varepsilon \leqslant f(x_p) - f(x_q) \leqslant k\varepsilon$ which proves the convergence of $\{f(x_n)\}$.

Conversely, if we choose any sequence $\{f_n(x)\}$ of additive non-negative functionals, the functional f_n being defined in $\mathfrak{E}_n$ (where $n=1,2,\ldots$) subject to the condition (3), then the formula (2) determines an additive non-negative functional $f(x)$ in $\mathfrak{E}$. We thus obtain the most general additive non-negative functional $f(x)$ defined in $\mathfrak{E}$, and by what has been established in the preceding §, the most general Lebesgue integral for all functions bounded and measurable $(\mathfrak{B})$.

The set $\mathfrak{E}_n$ may be interpreted as the set of all function of n variables $\vartheta_1,\ldots,\vartheta_n$ which are defined and continuous in the sphere $\vartheta_1^2+\ldots+\vartheta_n^2\leqslant 1$. It is known that the most general additive and non-negative functional defined in $\mathfrak{E}_n$ may be represented by a Stjeltjes integral.

These general considerations will now be illustrated by the following example. Suppose that the functionals f_n are given by the formula

$$(4)\qquad f_n(x)=\int\limits_{\vartheta_1^2+\ldots+\vartheta_n^2\leqslant 1}\cdots\int x(\vartheta_1,\ldots,\vartheta_n,0,0,\ldots)\,\varphi_n(\vartheta_1,\ldots,\vartheta_n)\,d\vartheta_1\ldots d\vartheta_n$$

for $x\in\mathfrak{E}_n$, where φ_n denotes a fixed non-negative function integrable in the sphere $\vartheta_1^2+\ldots+\vartheta_n^2\leqslant 1$. The condition (3) may be written in the form

$$\varphi_n(\vartheta_1,\ldots,\vartheta_n)=\int\limits_{-\sqrt{1-\vartheta_1^2-\ldots-\vartheta_n^2}}^{+\sqrt{1-\vartheta_1^2-\ldots-\vartheta_n^2}}\varphi_{n+1}(\vartheta_1,\ldots,\vartheta_n,\vartheta_{n+1})\,d\vartheta_{n+1}.$$

To satisfy this condition, we may put, for instance, $\varphi_1=1/2$ and $\varphi_{n+1}=\varphi_n/2\sqrt{1-\vartheta_1^2-\ldots-\vartheta_n^2}$ for $n\geqslant 1$. We thus obtain

$$(5)\qquad \varphi_n(\vartheta_1,\ldots,\vartheta_n)=\frac{1}{2^n\sqrt{1-\vartheta_1^2}\ldots\sqrt{1-\vartheta_1^2-\ldots-\vartheta_n^2}}.$$

Let x be an arbitrary function bounded and continuous in $\boldsymbol{H}$. We write again $x_n=x(\vartheta_1,\ldots,\vartheta_n,0,0,\ldots)$. If $|x|\leqslant M$, where M is a constant, then $\lim\limits_n x_n=x$, $|x_n|\leqslant M$.

Now let F be an $\mathfrak{L}$-integral which for functions belonging to $\mathfrak{E}$ coincides with the functional f subject to (2). We then have $F(x)=\lim_n F(x_n)=\lim_n f_n(x_n)$. If further f_n is represented by the formula (4), then

$$F(x)=\lim_n \int\limits_{\vartheta_1^2+\ldots+\vartheta_n^2\leqslant 1}\ldots\int x(\vartheta_1,\ldots,\vartheta_n,0,0,\ldots)\,\varphi_n(\vartheta_1,\ldots,\vartheta_n)\,d\vartheta_1\ldots d\vartheta_n$$

and, in particular, if φ_n is given by (5),

$$F(x)=\lim_n \int\limits_{\vartheta_1^2+\ldots+\vartheta_n^2\leqslant 1}\ldots\int x(\vartheta_1,\ldots,\vartheta_n,0,0,\ldots)\,\frac{d\vartheta_1\ldots d\vartheta_n}{2^n\sqrt{1-\vartheta_1^2}\ldots\sqrt{1-\vartheta_1^2-\ldots-\vartheta_{n-1}^2}}.$$

This formula defines explicitly a certain $\mathfrak{L}$-integral for all functions bounded and continuous in $\boldsymbol{H}$.

The above considerations may be extended to certain spaces of the type (B) (cf. S. Banach [I, Chap. V]), e. g. the spaces $l^{(p)}$, $L^{(p)}$ with $p>1$.