On measures in independent fields*

Edited by S. Hartman

Among the papers left by Banach was found the incomplete Polish manuscript of this paper, written in 1940. § 1 is almost literally translated from the manuscript. The details of farther reasonings were elaborated by S. Hartman, who also supplied the paper with Appendices and adapted it for print, with some help of Henry Helson.

§ 1. Let $T$ be an arbitrary space. A family $\mathcal{R}$ of fields (1) of subsets of $T$ is said to be a family of independent fields if any finite number of non-empty sets, belonging to different fields of $\mathcal{R}$, has a non-empty intersection. That is, $\mathcal{R}$ is an independent family if the conditions $0 \neq H_i \in A_i \in \mathcal{R}$ and $A_i \neq A_j$ for $i \neq j$ $(i, j = 1, \ldots, n)$ always imply

$$\prod_{i=1}^{n} H_i \neq 0.$$

The family $\mathcal{R}$ is called a family of denumerably independent fields if any sequence of non-empty sets, belonging to different fields of $\mathcal{R}$, has a non-empty intersection; i.e. if $0 \neq H_i \in A_i \in \mathcal{R}$ and $A_i \neq A_j$ for $i \neq j$ $(i, j = 1, 2, \ldots)$ always imply $\prod_{i=1}^{\infty} H_i \neq 0$.

The concept of independence of fields of sets was introduced by Marczewski (2), who also proved the following theorem (3):

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* Commenté sur p. 363.

(1) The class $A$ of subsets of a space $T$ is called a field if $A$ contains with any set its complement and with any finite number of sets their sum. The field $A$ is a Borel field if the sum of any denumerable number of sets of $A$ belongs to $A$.


(3) Ibidem, Théorème II, p. 126-127. For the proof of this theorem see Marczewski [6].
Let $\mathcal{R}$ be a family of independent fields with a measure $^{(1)} \mu$ defined in each field $A \in \mathcal{R}$, and let $U(\mathcal{R})$ be the smallest field containing all the fields of $\mathcal{R}$. Then a measure $\mu^*$ can be defined in $U(\mathcal{R})$ with the following properties:

(I) $\mu^*(H) = \mu(H)$ if $H \in A \in \mathcal{R}$,

(II) $\mu^* \left( \bigcap_{i=1}^{n} H_i \right) = \prod_{i=1}^{n} \mu^*(H_i)$ if $H_i \in A_i \in \mathcal{R}$, $A_r \neq A_s$ for $r \neq s$ and $n$ a natural number not greater than the power of $\mathcal{R}$ (which can be finite)$^{(2)}$.

Marczewski has asked whether the following theorem is true $^{(3)}$:

**Theorem 1.** Let $\mathcal{R}$ be a family of denumerably independent Borel fields with a denumerably additive measure $\mu$ defined in each field $A \in \mathcal{R}$; then a denumerably additive measure $\mu^*$ can be defined in the smallest Borel field containing all the fields of $\mathcal{R}$, such that:

1. $\mu^*(H) = \mu(H)$ if $H \in A \in \mathcal{R}$,

2. $\mu^* \left( \bigcap_{i=1}^{\infty} H_i \right) = \prod_{i=1}^{\infty} \mu^*(H_i)$ if $H_i \in A_i \in \mathcal{R}$, and $A_i \neq A_j$ for $i \neq j$

$(i, j = 1, 2, \ldots)$.

The object of this paper is to answer the question affirmatively.

Theorem 1 was proved by Marczewski in the special case that every field $A \in \mathcal{R}$ contains just four sets $^{(4)}$, viz. a set $H$, its complement, the empty set, and $T$. Then evidently every $A \in \mathcal{R}$ is a Borel field and any measure $\mu$ defined in $A$ is denumerably additive. The theorem was enunciated by P. Lévy in another special case, namely when $\mathcal{R}$ consists of two fields and the measures have a special form $^{(5)}$.

**§ 2.** The smallest (finitely additive) field containing all the fields of $\mathcal{R}$ will be denoted, as above, by $U(\mathcal{R})$. To prove Theorem 1 it is enough

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$^{(1)}$ A measure $\mu$ in a field $A$ is a real function $\mu(H) \geq 0$ defined for every set $H \in A$, such that $\mu(T) = 1$ and $\mu(H_1 + H_2) = \mu(H_1) + \mu(H_2)$ for any disjoint $H_1$, $H_2 \in A$. The measure is denumerably additive if $\mu \left( \sum_{i=1}^{\infty} H_i \right) = \sum_{i=1}^{\infty} \mu(H_i)$ for disjoint $H_1, H_2, \ldots \in A$.

$^{(2)}$ Fields $A_i$ for which condition (II) holds are said to be stochastically independent with respect to the measure $\mu^*$.

$^{(3)}$ loco cit.\textsuperscript{2}, Théorème II\textsuperscript{3}, especially p. 127.


to define a measure $\mu^*$ in $U(\mathcal{K})$ satisfying (1), (2) and the following condition:

(3) \quad \text{If } H_i \in U(\mathcal{K}) \text{ (} i = 1, 2, \ldots \text{) are disjoint and if } \sum_{i=1}^{\infty} H_i \in U(\mathcal{K}), \text{ then}
\quad \mu^* \left( \sum_{i=1}^{\infty} H_i \right) = \sum_{i=1}^{\infty} \mu^*(H_i).

For then it is known that $\mu^*$ can be extended to a denumerably additive measure on the smallest Borel field over $U(\mathcal{K})$, i.e. the smallest Borel field containing all the fields of $\mathcal{K}$.

Moreover, (3) can be replaced by the equivalent condition:

(4) \quad \text{If } H_i \in U(\mathcal{K}), \ H_i \supset H_{i+1} \ (i = 1, 2, \ldots) \text{ and } \lim_{i \to \infty} \mu^*(H_i) > 0, \text{ then}
\quad \prod_{i=1}^{\infty} H_i \neq 0.

Let $F[U(\mathcal{K})]$ be the family of all real functions defined for $t \in T$, assuming only a finite number of values, each value being assumed on a set belonging to $U(\mathcal{K})$. Every function $y \in F[U(\mathcal{K})]$ can be written in the following form (see Appendix I):

(5) \quad y(t) = \sum a_{k_1 \ldots k_m} \prod_{j=1}^{m} z_{j_k}(t).

Here the $z_{j_k}(t)$ are the characteristic functions of non-empty sets $Z_{j_k}$, which belong to different fields $A_j \in \mathcal{K}$ for different $j$, and which are disjoint in $k$ for any field $j$. That is:

(a) \quad Z_{j_k} \neq 0,
(b) \quad Z_{j_k} \in A_j \in \mathcal{K}, \ A_i \neq A_j \text{ for } i \neq j,
(c) \quad Z_{j_k} \cdot Z_{j_l} = 0, \text{ for } k \neq l \text{ and } j = 1, 2, \ldots, m.

The coefficients $a_{k_1 \ldots k_m}$ are real numbers, and the summation is extended over all systems $k_1 \ldots k_m$ which satisfy the conditions $1 \leq k_j \leq r_j$ ($j = 1, \ldots, m$).

§ 3. Uniformization. Let $y_1$ and $y_2$ be two (not necessarily different) functions from $F[U(\mathcal{K})]$. They are said to be uniformized when they are written

(6) \quad y_1(t) = \sum a_{k_1 \ldots k_m}^{(1)} \prod_{j=1}^{m} z_{j_k}(t),
(7) \quad y_2(t) = \sum a_{k_1 \ldots k_m}^{(2)} \prod_{j=1}^{m} z_{j_k}(t),
where the \( \varepsilon_{jk}(t) \) are the characteristic functions of sets \( Z_{jk} \) which satisfy conditions (a)-(c). The summations in (6) and (7) are extended over the same systems of indices \( k_1 \ldots k_m (1 \leq k_j \leq r_j) \). Thus the right sides of (6) and (7) can differ only in the coefficients \( a_{k_1 \ldots k_m} \).

**Refinement.** Let \( y \) be a function given in the form (5), and let \( B_1, B_2, \ldots, B_p \ (p \geq m) \) be distinct fields of \( \mathcal{R} \), among which occur the \( A_j \). For each \( j \) suppose \( s_j \) non-empty disjoint sets \( U_{ji} \in B_j \) are given such that if \( B_j \) is identical with some \( A_{i_1} \), then every \( Z_{jk} \) is the sum of some of the \( U_{ji} \), and if \( B_j \) is different from all the \( A_i \), the sum of the \( U_{ji} \) is \( T \).

Then (5) can be transformed (see Appendix II, 1°) into the following:

(8) \[ y(t) = \sum_{i_1 \ldots i_p} b_{i_1 \ldots i_p} \prod_{j=1}^{p} u_{ji_j}(t) \quad (1 \leq i_j \leq s_j), \]

where \( u_{ji_j}(t) \) is the characteristic function of \( U_{ji} \). We call (8) a refinement of (5) by the sets \( U_{ji} \).

For two functions (not necessarily different) from \( F[U(\mathcal{R})] \) given in the form (5), there always exists (see Appendix III) a system of sets \( U_{ji} \) by which both functions can be refined. Such a common refinement uniformizes the functions. Hence any two functions of \( F[U(\mathcal{R})] \) can be uniformized.

**Denumerable uniformization.** If \( y_n \) is a sequence of functions belonging to \( F[U(\mathcal{R})] \), in general no uniformization is possible for all \( y_n \) in the sense defined above.

However, the following representation can always be reached (see Appendix IV):

(9) \[ y_n(t) = \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m}^{(n)} \prod_{j=1}^{m} \varepsilon_{jk_j}(t) \quad \text{for} \quad n = 1, 2, \ldots, \]

where \( \varepsilon_{jk_j}^{(n)} \) is the characteristic function of a non-empty set \( Z_{jk_j} \in B_j \in \mathcal{R} \), with the sequence \( \{B_j\} \) (generally infinite) containing no field more than once. The numbering of the \( B_j \) is determined simultaneously for all \( n \). Further, as usual, \( Z_{jk_j} \cdot Z_{js_j} = 0 \) for \( r \neq s \), and the summation in (9) is taken over all systems \( k_1 \ldots k_m \) such that \( 1 \leq k_j \leq r_j^{(n)} \). In general, the sequence \( m_n \) is unbounded as \( n \) increases. The sequence \( y_n \), given in the form (9), is said to be *denumerably uniformized.*

**§ 4. Lemma 1.** Let two uniformized functions \( y_1 \) and \( y_2 \) of \( F[U(\mathcal{R})] \) be given:

\[ y_1(t) = \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m}^{(1)} \prod_{j=1}^{m} \varepsilon_{jk_j}(t), \quad y_2(t) = \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m}^{(2)} \prod_{j=1}^{m} \varepsilon_{jk_j}(t). \]
If for every $t$ we have $y_1(t) = y_2(t)$ or $y_1(t) \geq y_2(t)$, then for every set of indices we have $a_{k_1 \ldots k_m}^{(1)} = a_{k_1 \ldots k_m}^{(2)}$ or $a_{k_1 \ldots k_m}^{(1)} \geq a_{k_1 \ldots k_m}^{(2)}$ respectively.

**Proof.** If we set $a_{k_1 \ldots k_m}^{(1)} - a_{k_1 \ldots k_m}^{(2)} = a_{k_1 \ldots k_m}$, then

$$\sum_{j=1}^{m} a_{k_1 \ldots k_m} \prod_{j=1}^{m} z_{jk_j}(t) = 0 \text{ or } \geq 0, \text{ resp.}, \text{ for all } t.$$

From conditions (a) and (b) and from the independence of the fields of $R$ it follows that for every system of indices $\sigma_1, \ldots, \sigma_m$ there is a $t \in T$ for which $\prod_{j=1}^{m} z_{j\sigma_j}(t) = 1$.

From (\gamma), $\prod_{j=1}^{m} z_{jk_j}(t) = 0$ for the same $t$ and any other system $k_1 \ldots k_m$.

So for this $t$,

$$\sum_{j=1}^{m} a_{k_1 \ldots k_m} \prod_{j=1}^{m} z_{jk_j}(t) = a_{\sigma_1 \ldots \sigma_m},$$

and $a_{\sigma_1 \ldots \sigma_m} = 0$ or $\geq 0$, according as $y_1(t) = y_2(t)$ or $y_1(t) \geq y_2(t)$ for all $t$.

Since the set $\sigma_1 \ldots \sigma_m$ was arbitrary, the lemma is proved.

**Remark.** Only the finite independence of the fields of $R$ was used in the proof; their denumerable independence will be used later.

**Lemma 2.** If $y(t) \geq 0$, or $= 0$, or $\leq 0$ for all $t$, then for every system of indices, $a_{k_1 \ldots k_m} \geq 0$, or $= 0$, or $\leq 0$ respectively.

This is an immediate consequence of Lemma 1.

**§ 5.** We now introduce three operations on the functions $y \in F[U(R)]$, called integration, contraction and separation.

**Integration.** If $y$ is given by (5), write formally

$$\int_{T} y(t) dt = \sum_{j=1}^{m} a_{k_1 \ldots k_m} \prod_{j=1}^{m} \mu z_{jk_j}.$$

We show that the integral does not depend on the particular representation of $y$ (always, of course, of the same type as (5)). Indeed, suppose

$$y(t) = \sum_{j=1}^{m} b_{k_1 \ldots k_p} \prod_{j=1}^{p} v_{jk_j}(t),$$

where the $v_{jk}(t)$ are the characteristic functions of sets $V_{jk}$. Let $U_{\bar{m}}$ be a system of sets which refines both representations, yielding

$$y(t) = \sum_{j=1}^{m} c_{k_1 \ldots k_q}^{(1)} \prod_{j=1}^{q} u_{jk_j}(t), \quad y(t) = \sum_{t=1}^{\bar{m}} c_{k_1 \ldots k_q}^{(2)} \prod_{t=1}^{q} u_{jk_j}(t)$$
respectively, where \( u_{jk} \) is the characteristic function of \( U_{jk} \). By Lemma 1, \( c^{(1)}_{k_1...k_q} = c^{(2)}_{k_1...k_q} \) for every set of indices; by Appendix II, \( 2^n \) the integral is not changed by refinement. That is,

\[
\sum a_{k_1...k_m} \prod_{j=1}^m \mu(Z_{jk}) = \sum c^{(1)}_{k_1...k_q} \prod_{j=1}^a \mu(U_{jk})
\]

\[
= \sum c^{(2)}_{k_1...k_q} \prod_{j=1}^a \mu(U_{jk}) = \sum b_{k_1...k_q} \prod_{j=1}^p \mu(V_{jk}),
\]

which establishes the invariance of the definition.

Let \( y_1 \) and \( y_2 \) be functions from \( F[U(\mathbb{R})] \), with uniformized representations

\[
y_1(t) = \sum a^{(1)}_{k_1...k_m} \prod_{j=1}^m u_{jk}(t), \quad y_2(t) = \sum a^{(2)}_{k_1...k_m} \prod_{j=1}^m u_{jk}(t).
\]

It is immediate that

\[
\int_T (y_1(t) + y_2(t)) dt = \int_T y_1(t) dt + \int_T y_2(t) dt,
\]

\[
\int_T a y(t) dt = a \int_T y(t) dt
\]

for any number \( a \). By Lemma 2,

\[
\int_T y(t) dt \geq 0 \quad \text{if} \quad y(t) \geq 0 \text{ for all } t.
\]

Denote by \( F(A) \) the subset of \( F[U(\mathbb{R})] \) composed of all functions which assume each of their values in a set belonging to \( A \in \mathbb{R} \), and let \( y(t) \in F(A) \). The representation (5) becomes then

\[
y(t) = \sum_{k=1}^m a_k z_k(t),
\]

and the corresponding integral is \( \sum_{k=1}^m a_k \mu(Z_k) \); this integral is identical with that in customary sense engendered by the measure \( \mu \). Hence follows

\[
\int_T 1 dt = \mu(T) = 1,
\]

the function \( y(t) = c \) being contained in all sets \( F(A) \).

\(^{(1)}\) Only finite additivity of the measure \( \mu \) is assumed there. Denumerable additivity will be required later.
From (11) and (13) it follows that

\[ (15) \quad \text{if } y_1(t) \geq y_2(t) \text{ for all } t, \text{ then } \int_T y_1(t) \, dt \geq \int_T y_2(t) \, dt. \]

Finally suppose \( U \in \mathcal{A} \subseteq \mathbb{R} \), \( V \in \mathcal{B} \subseteq \mathbb{R} \), and let \( u, v \) be the characteristic functions of \( U, V \) resp. By the definition of the integral,

\[ (16) \quad \int_T u(t) \cdot v(t) \, dt = \mu(U) \cdot \mu(V). \]

**Contraction.** Given \( y(t) = \sum_{k_1 \ldots k_m} \mu \prod_{j=1}^{m} z_{j k_j}(t) \), define:

\[ (17) \quad W(y, t) = \sum_{k_1 \ldots k_m} \mu \prod_{j=2}^{m} z_{j k_j}(t), \quad \text{if } m \geq 2, \]

\[ (18) \quad W(y, t) = y(t) \quad \text{if } m = 1. \]

We call (17) the *contraction* of \( y \) with respect to the field \( \mathcal{A}_1 \). The result of the operation evidently depends on the choice of \( \mathcal{A}_1 \); however, a proof entirely analogous to that given for integration and in Appendix II shows that \( W(y, t) \) is not changed by refinement of \( y \), and it follows that the definition does not depend on the representation of \( y \), once \( \mathcal{A}_1 \) has been fixed. Exactly as for the integral one proves that contraction by a given field is a linear operation, and that if \( y(t) \geq 0 \) for all \( t \), the same is true of \( W(y, t) \). Hence

\[ (19) \quad \text{if } y_1(t) \geq y_2(t) \text{ for all } t, \text{ then } W(y_1, t) \geq W(y_2, t) \text{ for all } t, \]

where contraction is performed with respect to the same field.

**Separation.** Again suppose \( y(t) = \sum_{k_1 \ldots k_m} \mu \prod_{j=1}^{m} z_{j k_j}(t) \). For any \( t_1 \in T \) define

\[ (20) \quad S(y, t_1, t) = \sum_{k_1 \ldots k_m} \mu \prod_{j=2}^{m} z_{j k_j}(t_1) \prod_{j=2}^{m} z_{j k_j}(t). \]

We call this operation *separation* performed at \( t_1 \) with respect to the field \( \mathcal{A}_1 \). Again it is easy to show that \( S(y, t_1, t) \) depends only on \( y, t_1, t \) and \( \mathcal{A}_1 \), and not on the representation of \( y \) once \( \mathcal{A}_1 \) has been fixed. For fixed \( t_1 \) and \( \mathcal{A}_1 \), separation is a linear operation; and if \( y \) is non-negative, so is \( S(y, t_1, t) \). Hence

\[ (21) \quad \text{if } y_1(t) \geq y_2(t) \text{ for all } t, \text{ then } S(y_1, t_1, t) \geq S(y_2, t_1, t) \text{ for all } t, t_1, \]

where separation is performed with respect to the same field.
Lemma 3. If \( y \) is of the form (5), then
\[
\int \limits_{\mathcal{T}} S(y, t_1, t) \, dt = W(y, t_1) \quad \text{and} \quad \int \limits_{\mathcal{T}} W(y, t) \, dt = \int \limits_{\mathcal{T}} y(t) \, dt.
\]

These relations are evident on writing out the integrals explicitly.

§ 6. Measure. Let \( E \) belong to \( U(\mathcal{R}) \), with characteristic function \( y \). Define
\[
\mu^*(E) = \int \limits_{\mathcal{T}} y(t) \, dt.
\]

It follows from (11) and (13) that \( \mu^* \) is a (finitely additive) measure, and by the definition of the integral we have \( \mu^*(E) = \mu(E) \) for \( E \in \mathcal{A} \in \mathcal{R} \). By (16) the fields \( \mathcal{A} \in \mathcal{R} \) are stochastically independent with respect to \( \mu^* \). So \( \mu^* \) satisfies the conditions of the theorem of Marczewski. Since only the hypotheses of that theorem have been used, we have proved it on the way to the main result.

§ 7. Lemma 4. Let \( y \) be of form (5), \( a \) a real number, and \( t_1, \ldots, t_m \) points of \( \mathcal{T} \) such that
\[
Y(t_1, \ldots, t_m) \geq \sum_{k=1}^{m} \frac{a_{s_1} \ldots s_m}{d} \prod_{j=1}^{m} z_{jk}(t_j) \geq a.
\]

Then there are sets \( H_j \in \mathcal{A}_j \) \( (j = 1, \ldots, m) \) for which
\[
t_j \in H_j,
\]
\[
y(t) \geq a \quad \text{for all} \quad t \in \prod_{j=1}^{m} H_j.
\]

Proof. First assume that \( t_j \in \sum_{k=1}^{r_j} Z_{jk} \) for each \( j \); that is, there are indices \( \sigma_1, \ldots, \sigma_m \) for which \( t_j \in Z_{j\sigma_j} \). Let \( H_j = Z_{j\sigma_j} \). Evidently \( H_j \in \mathcal{A}_j \), so it remains to prove (24). Notice that \( k_j \neq \sigma_j \) implies \( t_j \notin Z_{j\sigma_j} \), or \( z_{jk}(t_j) = 0 \), so
\[
Y(t_1, \ldots, t_m) = a_{\sigma_1} \ldots \sigma_m;
\]
hence \( a_{\sigma_1} \ldots \sigma_m \geq a \). Now if \( t \in \prod_{j=1}^{m} H_j \), we have also
\[
y(t) = a_{\sigma_1} \ldots \sigma_m \geq a.
\]

Now suppose \( t_j \notin \sum_{k=1}^{r_j} Z_{jk} \) for at least one \( j \), say for \( j = s_1, \ldots, s_p \) \( (1 \leq p \leq m) \). For each such \( j \), \( z_{jk}(t_j) = 0 \) \( (k = 1, \ldots, r_j) \), so that \( \prod_{j=1}^{m} z_{jk}(t_j) = 0 \) for any indices \( k_1, \ldots, k_m \). Hence \( Y(t_1, \ldots, t_m) = 0 \) and \( a \leq 0 \). Set \( H_j = T - \sum_{k=1}^{r_j} Z_{jk} \) for \( j = s_1, \ldots, s_p \), and \( H_j = \sum_{k=1}^{r_j} Z_{jk} \) for other \( j \). Then
$t_j \in H_j \in A_j$. If $t \in \prod_{k=1}^{m} H_j$, then $t \not\in Z_{jk} (k = 1, \ldots, r_j)$ for at least one $j$ (since $p \geq 1$), from which $\prod_{j=1}^{m} z_{j,k_j}(t) = 0$ for any choice of the $k_j$. Hence $y(t) = 0 \geq a$, and (24) holds.

**Lemma 5.** Let $y_n(t) = \sum a_{k_1 \ldots k_n}^{(n)} \prod_{j=1}^{m} z_{j,k_j}(t)$ be a sequence of denumerably uniformized functions from $F[U(\aleph)]$; suppose there is a finite or infinite sequence $t_1, t_2, \ldots$ of elements of $T$, such that for each $n$

$$Y_n(t_1, \ldots, t_{m_n}) \geq a.$$ 

Then there is some $\theta \in T$ for which $y_n(\theta) \geq a$ ($n = 1, 2, \ldots$).

**Proof.** By Lemma 4, for each $n$ there are sets $H_j^n \in A_j (j = 1, \ldots, m_n)$ with the properties:

$$t_j \in H_j^n \quad \text{for} \quad j = 1, \ldots, m_n,$$

$$y_n(t) \geq a \quad \text{for} \quad t \in \prod_{j=1}^{m_n} H_j^n.$$

For $j > m_n$ set $H_j^n = T$ and define $W_j = \prod_{j=1}^{\infty} H_j^n$. Each $W_j$ is non-empty, since $t_j \in W_j$; and because the $A_j \in \aleph$ are Borel fields (the first use of this hypothesis), $W_j \in A_j$. Now set $H = \prod_{j=1}^{\infty} W_j$. This intersection is non-empty, because the $A_j \in \aleph$ are denumerably independent (this is the only use of the hypothesis in the proof!). But if $\theta \in H$, then $\theta \in \prod_{j=1}^{m_n} H_j^n$ and $y_n(\theta) \geq a$ for all $n$. So the lemma is proved.

**§ 8. Proof of Theorem 1.** It only remains to prove that $\mu^*$ satisfies condition (4). Suppose

$$H_n \in U(K), \quad H_n \supset H_{n+1}, \quad \mu^*(H_n) \geq a > 0, \quad n = 1, 2, \ldots$$

Let $y_n$ be the characteristic function of $H_n$. Then $y_n \in F[U(K)]$, $\int \int y_n(t) \, dt \geq a$, and for all $t$

$$y_n(t) \geq y_{n+1}(t) \quad (n = 1, 2, \ldots).$$

(25)

We take the $y_n$ denumerably uniformized:

$$y_n(t) = \sum a_{k_1 \ldots k_n}^{(n)} \prod_{j=1}^{m_n} z_{j,k_j}(t).$$
Then
\begin{equation}
\label{eq:26}
m_{n+1} \geq m_n \quad (n = 1, 2, \ldots).
\end{equation}

By (19), for all \( t \)
\begin{equation}
\label{eq:27}
W(y_n, t) \geq W(y_{n+1}, t) \quad (n = 1, 2, \ldots),
\end{equation}
and by Lemma 3
\begin{equation}
\label{eq:28}
\int_T W(y_n, t) dt \geq a \quad (n = 1, 2, \ldots).
\end{equation}

Now every \( W(y_n, t) \) belongs to \( F(A_1) \). But \( \mu^* \) is in \( A_1 \) a denumerably additive measure (this is the first use of the hypothesis), and the integral of \( y_n(t) \) over \( T \) (in the sense given p. 279) is an integral generated by \( \mu^* \) in the Lebesgue sense. Hence, applying (28),
\begin{equation}
\label{eq:29}
\int_T \lim_{n \to \infty} W(y_n, t) dt = \lim_{n \to \infty} \int_T W(y_n, t) dt \geq a.
\end{equation}

By (27) and (29) there is an element \( t_1 \) for which
\begin{equation}
\label{eq:30}
W(y_n, t_1) \geq a \quad (n = 1, 2, \ldots).
\end{equation}

Now set \( y_{n,1}(t) = S(y_n, t_1, t) \). These functions are already denumerably uniformized. By (21), \( y_{n,1}(t) \geq y_{n+1,1}(t) \) for all \( t \); by (30) and Lemma 3
\begin{equation}
\int_T y_{n,1}(t) dt \geq a \quad (n = 1, 2, \ldots).
\end{equation}

So the \( y_{n,1} \) are like the \( y_n \), and there is a \( t_2 \) for which
\begin{equation}
W(y_{n,1}, t_2) \geq a \quad (n = 1, 2, \ldots),
\end{equation}
where contraction is performed with respect to \( A_2 \). Setting \( y_{n,2}(t) = S(y_{n,1}, t_2, t) \), we have \( y_{n,2}(t) \geq y_{n+1,2}(t) \) for all \( t \) and
\begin{equation}
\int_T y_{n,2}(t) dt \geq a \quad (n = 1, 2, \ldots).
\end{equation}

This procedure can be repeated indefinitely by setting
\[ y_{n,k+1}(t) = S(y_{n,k}, t_{k+1}, t). \]

If \( m_n = l \) for some \( n \) (or several, or infinitely many \( n \)), say for \( n \) such that \( p \leq n \leq r \), we have for these \( n \) after the \((l-1)\)th step
\begin{equation}
\label{eq:31}
y_{n,l-1}(t) = \sum a_{k_1 \ldots k_l}^{(n)} z_{k_1}^{(n)}(t_1) \prod_{j=1}^{l-1} z_{k_j}^{(n)}(t_j).
\end{equation}
The functions \( y_{n,l-1} \) with \( p \leq n \leq r \) each belongs to \( F(A_l) \). Contraction with respect to \( A_l \) leaves by (18) the \( y_{n,l-1} \) invariant. Having fixed \( l \), the \( W(y_{n,l-1}, t) \) is a non-increasing sequence in the index \( n \), and for every \( n \)

\[ \int T W(y_{n,l-1}, t) \, dt \geq a. \]

It follows that there is a \( t_l \) such that

\[ W(y_{n,l-1}, t_l) \geq a \quad (n = 1, 2, \ldots) \]

and in particular

\[ y_{n,l-1}(t_l) \geq a \quad \text{for} \quad p \leq n \leq r. \tag{32} \]

For these values of \( n \), however, the left side of (32) is by (31), on one hand, the function \( y_{n,l}(t) = S(y_{n,l-1}, t, t) \), which is a constant, and, on the other hand, identical with \( Y_n(t_1, \ldots, t_m) \).

In case \( m_n = l \) for all \( n \geq p \), the process is finished; otherwise continue with those \( y_{n,l}(t) \) for which \( m_n > l \), in other words, for which \( n > r \). Finally, a (possibly finite) sequence of elements \( t_1, t_2, \ldots \) is at hand, which with the \( y_n \) satisfy the hypothesis of Lemma 5.

So there is a \( \theta \in T \) for which \( y_n(\theta) \geq a \quad (n = 1, 2, \ldots) \). Since \( a > 0 \) and the \( y_n \) are characteristic functions, \( y_n(\theta) = 1 \), so that \( \theta \in H_n \quad (n = 1, 2, \ldots) \).

Hence \( \prod_{n=1}^{\infty} H_n \neq 0 \), (4) is shown, and Theorem 1 is proved.

### APPENDIX I

**Representation of functions**

Let \( H_v \ (v = 1, \ldots, p) \) be sets belonging to \( U(\mathfrak{K}) \). For every \( v \)

\[ H_v = \sum_{i=1}^{n_v} \prod_{j=1}^{m_{v,i}} G_{ij}^{v}, \tag{1} \]

where each set \( G_{ij}^{v} \) belongs to a field \( A_{ij}^{v} \in \mathfrak{K} \), with \( A_{ir}^{v} \neq A_{is}^{v} \) if \( r \neq s \). Renumber the fields \( A_{ij}^{v} \) in simple order: \( A_1, \ldots, A_m \), where distinct indices belong to distinct fields. In each intersection of (1) write the factors \( G_{ij}^{v} \) in the order of the indices of the fields to which they belong, supplying a \( T \) as the \( r^{th} \) factor if no \( G_{ij}^{v} \) belongs to \( A_r \ (v = 1, \ldots, m) \). By the independence of the fields, no set except 0 and \( T \) can belong to more
than one field, so this rearrangement can be carried out without ambiguity. By renaming the reordered factors we have a representation

\[ H_v = \sum_{i=1}^{n_v} \prod_{j=1}^{m} H_{ij}, \]

where \( H_{ij} \in A_{j} \in S \) (\( A_r \neq A_s \) for \( r \neq s \)).

Now for each \( j \) put the sets \( H_{ij} \) in a simple series \( H_{jk} \) \((k = 1, \ldots, s_j; H_{jk} \neq H_{jl} \) for \( k \neq l \)). For each \( j \) and every system \( \sigma_1 \ldots \sigma_{s_j} \) composed of zeros and ones, form the product

\[ \prod_{k=1}^{s_j} (-1)^{\sigma_k} H_{jk}, \]

where \(-H_{jk}\) means \( T - H_{jk}\).

Suppose \( r_j \) of these intersections are non-void; call them \( Z_{jk} \) \((k = 1, \ldots, r_j)\). Evidently \( Z_{jk} \in A_j \), \( Z_{jk} \cdot Z_{jl} = 0 \) for \( k \neq l \), and every \( H_{ij}^r \) is the sum of some of the sets \( Z_{j1}, \ldots, Z_{jr_j} \). So there are numbers \( \beta_{ijk}^r \) (each 0 or 1) such that

\[ H_{ij}^r = \sum_{k=1}^{r_j} \beta_{ijk}^r Z_{jk}. \]

Setting this in (2) we obtain

\[ H_v = \sum_{i=1}^{n_v} \left[ \sum_{k=1}^{r_k} \beta_{ik1} \cdot \ldots \cdot \beta_{imk} \prod_{j=1}^{m} Z_{jk} \right], \]

where the outer summation is taken over all systems \( k_1 \ldots k_m \) \((1 \leq k \leq r_j)\).

Setting \( a_{k_1 \ldots k_m}^r \) equal to the smaller value of 1 and \( \sum_{i=1}^{n_v} \beta_{i1k_1} \cdot \ldots \cdot \beta_{imk_m}^r \), we get

\[ H_v = \sum a_{k_1 \ldots k_m}^r \prod_{j=1}^{m} Z_{jk}, \]

The intersections in (3) are disjoint.

Let \( Z_{jk} \) be the characteristic function of \( Z_{jk} \), and \( y_r \) the characteristic function of \( H_v \). Then

\[ y_r(t) = \sum a_{k_1 \ldots k_m}^r \prod_{j=1}^{m} z_{jk}(t). \]
Now if $y \in F[U(\mathfrak{S})]$ assumes its values $a_1, \ldots, a_p$ on the sets $H_1, \ldots, H_p$ respectively, $y(t) = \sum_{r=1}^{p} a_r y_r(t)$, i.e.

$$y(t) = \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m} \prod_{j=1}^{m} \varepsilon_{j,k_j}(t),$$

where the summation is taken over systems $k_1 \ldots k_m$ ($1 \leq k_j \leq r_j$), and

$$a_{k_1 \ldots k_m} = \sum_{r=1}^{p} a_r a_{k_1 \ldots k_m}^r.$$

**APPENDIX II**

**Refinement**

1° Let the function $y$ be given in the form

$$y(t) = \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m} \prod_{j=1}^{m} \varepsilon_{j,k_j}(t),$$

where each $\varepsilon_{j,k}$ is the characteristic function of a non-empty set $Z_{j,k} \in A_j \in \mathfrak{S}$ ($1 \leq k \leq r_j$), with $A_r \neq A_s$ for $r \neq s$; and $Z_{j,k} \cdot Z_{j,l} = 0$ for any $j$, when $k \neq l$.

Let $B_1, B_2, \ldots, B_m, \ldots, B_p$ be distinct fields of $\mathfrak{S}$, such that $B_i = A_i$ for $i \leq m$. Suppose further we are given sets $U_{ji} \in B$, non-empty, disjoint in $i$ for fixed $j$, and such that for $j \leq m$

$$Z_{j,k} = \sum_{i=1}^{s_j} \beta_{j,ki} U_{ji} \quad \text{(each } \beta_{j,ki} \text{ being either 0 or 1)},$$

and for $j > m$

$$T = \sum_{i=1}^{s_j} \beta_{j,li} U_{ji} \quad \text{(each } \beta_{j,li} = 1).$$

If $u_{ji}$ is the characteristic function of $U_{ji}$,

$$\sum_{i=1}^{s_j} \beta_{j,li} u_{ji}(t) = \varepsilon_{j,k}(t) \quad \text{when } j \leq m,$$

$$\sum_{i=1}^{s_j} \beta_{j,li} u_{ji}(t) = 1 \text{ for all } t, \quad \text{when } j > m.$$
Write unit factors in each product of (1) so the index \( j \) runs from 1 to \( p \), and then substitute formulas (4) and (5) for the functions \( z_{jkj} \) and for the unit factors respectively. Then we have

\[
y(t) = \sum a_{k_1 \ldots k_m} \prod_{j=1}^{p} \sum_{i_1=1}^{s_j} \beta_{jk_{1i_1}} \cdots \beta_{jk_{pi}} \prod_{j=1}^{p} u_{j_{ij}}(t),
\]

where the inner summation is taken over all sets \( i_1 \ldots i_p \) such that \( 1 \leq i_j \leq s_j \) for each \( j \), and the outer summation as before over sets \( k_1, \ldots, k_m \) such that \( 1 \leq k_j \leq r_j \). Changing the order of summation and writing

\[
b_{i_1 \ldots i_p} = \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m} \beta_{1k_{1i_1}} \cdots \beta_{pk_{pi}},
\]

we have

\[
y(t) = \sum b_{i_1 \ldots i_p} \prod_{j=1}^{p} u_{j_{ij}}(t).
\]

This representation is a refinement of (1) by the sets \( U_{ji} \).

2° We show that

\[
\sum a_{k_1 \ldots k_m} \prod_{j=1}^{m} \mu(Z_{jkj}) = \sum b_{i_1 \ldots i_p} \prod_{j=1}^{p} \mu(U_{j_{ij}}).
\]

Indeed, from (2)

\[
\mu(Z_{jk}) = \sum_{i=1}^{s_j} \beta_{jk_i} \mu(U_{ji}) \quad \text{for} \quad j \leq m,
\]

and from (3)

\[
1 = \sum_{i=1}^{s_j} \beta_{jk_i} \mu(U_{ji}) \quad \text{for} \quad j > m.
\]

By the same algebraic procedure as before, i.e. by writing \( p - m \) unit factors in each product on the left side of (8), and by substituting (9) and (10), we obtain

\[
\sum a_{k_1 \ldots k_m} \sum \beta_{1k_{1i_1}} \cdots \beta_{pk_{pi}} \prod_{j=1}^{p} \mu(U_{j_{ij}}),
\]

where the sums are as in (6). By changing the order of summation and using (7) the right side of (8) appears.
APPENDIX III

Common refinement

Let functions $y_1$ and $y_2$ of $F[U(\mathbb{R})]$ be given, not necessarily distinct:

\begin{align*}
(1) & \quad y_1(t) = \sum \alpha_{r_1 \ldots r_{m_1}}^{(1)} \prod_{j=1}^{m_1} z_{j_{r_j}}^{(1)}(t), \\
(2) & \quad y_2(t) = \sum \alpha_{r_2 \ldots r_{m_2}}^{(2)} \prod_{j=1}^{m_2} z_{j_{r_j}}^{(2)}(t).
\end{align*}

Here the $z_{j_{r_j}}^{(r)}$ are characteristic functions of sets $Z_{r_{j_k}}$ ($r = 1, 2; \quad j = 1, \ldots, m_r; \quad k = 1, \ldots, r_j^{(r)}$), such that $Z_{r_{j_k}} \neq 0$, $Z_{r_{j_k}} \in A_r^r \in \mathcal{R}$ ($A_r^r \neq A_s^s$ for $r \neq s$), and $Z_{r_{j_r}} \cdot Z_{r_{j_s}} = 0$ if $r \neq s$. There may or may not be fields $A_i^i$ identical with fields $A_j^j$. But form a series $B_1, \ldots, B_p$ out of the $A_i^i$ and the $A_j^j$ which includes each field just once, and for each of the $B_j$, renumber in a series $Z_{j_{s}}$ ($s = 1, \ldots, q_j$) all the sets $Z_{r_{j_k}}$ such that $A_r^r = B_j$. As in Appendix I form all the intersections

\begin{equation}
\prod_{s=1}^{q_j} (-1)^{s} Z_{j_{s}},
\end{equation}

where the $\sigma_s$ assume the values 0, 1; and number the non-void intersections $U_{j_{1}}, \ldots, U_{j_{s}}$. Evidently for each $j$, $U_{j_{i}} \in B_j$ and $U_{j_{i}} \cdot U_{j_{k}} = 0$ if $i \neq k$; each set $Z_{j_{s}}$ belonging to $B_j$ is the sum of some of the $U_{j_{k}}$, and for fixed $j$ the sum of the $U_{j_{i}}$ is $T$. Thus the sets $U_{j_{i}}$ yield a common refinement of (1) and (2).

APPENDIX IV

Denumerable uniformization

Let a sequence of functions from $F[U(\mathbb{R})]$ be given:

\begin{align*}
(1) & \quad y_n(t) = \sum b_{q_1 \ldots q_n}^{(m)} \prod_{j=1}^{p_n} u_{j_{r_j}}^{(m)}(t),
\end{align*}

where $u_{j_{r_j}}^{(m)}(t)$ is the characteristic function of a non-empty set

\[ U_{j_{r_j}}^n \in A_r^r \in \mathcal{R} \quad \text{for} \quad n = 1, 2, \ldots; \quad j = 1, \ldots, p_n; \quad i = 1, \ldots, s_j^{(n)}. \]

If $r \neq s$, $A_r^r \neq A_s^s$ and $U_{j_{r_j}}^n \cdot U_{j_{s}}^n = 0$. We shall transform (1) by induction so as to obtain a representation of the following kind:

\begin{equation}
\sum_{k=1}^{m_n} a_{k_1 \ldots k_{m_n}}^{(m)} \prod_{j=1}^{m_n} z_{j_{r_j}}^{(m)}(t),
\end{equation}
where \( z_{jk}^{(n)}(t) \) is the characteristic function of a set \( Z_{jk}^n \neq 0 \), \( Z_{jk}^n \in B_j \in \mathcal{R} \) for \( n = 1, 2, \ldots; j = 1, \ldots, m_n; \ i = 1, \ldots, r_j^{(n)} \); if \( r \neq s \), \( B_r \neq B_s \) and \( Z_{js}^n Z_{js}^n = 0 \). Here the sequence \( \{B_j\} \) does not involve the index \( n \).

Assume that \( y_1, \ldots, y_{n-1} \) are already expressed in the form (2), with the fields \( B_1, \ldots, B_{m_{n-1}} \) and the sets \( Z_{jk}^l \in B_j \) determined \( (l = 1, \ldots, n-1; j = 1, \ldots, m_{n-1}; k = 1, \ldots, r_j^{(l)}) \).

1° If a field \( A_j^n \) is identical with \( B_h \) for some \( h \leq m_{n-1} \), set \( U_{jk}^n = Z_{hk}^n \), and accordingly

\[
\begin{align*}
  u_{jk}^{(n)}(t) &= z_{jk}^{(n)}(t), \\
  s_j^{(n)} &= r_j^{(n)}.
\end{align*}
\]

2° If there are fields \( A_j^n \) which are not among the \( B_i \) \( (i = 1, \ldots, m_{n-1}) \), denote them by \( B_{m_{n-1}+1}, \ldots, B_{m_{n}} \), and write correspondingly:

\[
U_{jk}^n = Z_{m_{n-1}+1k}^n, \ldots, Z_{m_nk}^n,
\]

\[
\begin{align*}
  u_{jk}^{(n)}(t) &= z_{m_{n-1}+1k}^{(n)}(t), \ldots, z_{m_nk}^{(n)}(t) \quad (1 \leq k \leq s_j^{(n)}), \\
  s_j^{(n)} &= r_{m_{n-1}+1}^{(n)}, \ldots, r_{m_n}^{(n)}.
\end{align*}
\]

3° If some \( B_j \) \( (j \leq m_{n-1}) \) does not occur among the \( A_j^n \), set

\[
Z_{j1}^n = T, \quad z_{j1}^{(n)}(t) = 1 \quad \text{for all } t, \quad \text{and } r_j^{(n)} = 1.
\]

By this procedure each product

\[
(3) \quad \prod_{j=1}^{n} u_{j1}^{(n)}(t)
\]

of (1) is transformed into a product

\[
(4) \quad \prod_{j=1}^{m_n} z_{j1}^{(n)}(t),
\]

which differs from (3) only in the order of the factors and the presence of certain unit factors. Bearing in mind that \( r_j^{(n)} = 1 \) for the \( j \) considered in 3°, each \( y_{i_1 \ldots i_p}^{(n)} \) can be rewritten \( a_{k_1 \ldots k_{m_n}}^{(n)} \), and \( y_n \) has been reduced to form (2).