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avec l'article de A. Pełczyński
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SOME ASPECTS OF THE PRESENT THEORY OF BANACH SPACES

by

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INTRODUCTION

The purpose of this survey is to present some results in the fields of the theory of Banach spaces which were initiated in the monograph *Théorie des opérations linéaires*. The reader interested in the theory of functional analysis and the development of its particular chapters is referred to the Notes and Remarks in the monograph by Dunford and Schwartz [1], and to the Historical Remarks of Bourbaki [2] (*).

The extensive bibliography at the end of this survey concerns only the fields which are discussed here, but even in this respect it is not complete. Large bibliographies of various branches of functional analysis can be found in the following monographs: Dunford and Schwartz [1], Köthe [1], Lacey [1], Lindenstrauss and Tzafriri [1], Semadeni [1], Singer [1].

Banach's monograph *Théorie des opérations linéaires* is quoted in this survey as [B]. When writing, for instance, [B], Rem. V, § 2, we refer to "Remarques" to Chapter V, § 2 of the monograph.

Some recent information is contained in the section "Added in proof".

Notation and terminology. We attempt to adjust our notation to that which is now commonly used (e.g. in Dunford and Schwartz [1]) and which differs to some extent from the notation of Banach.

We use the symbols L^p, l^p, C, c, c_0, s instead of Banach's: $(L^{(p)})$, $(l^{(p)})$, etc. Also we write L^∞ and l^∞ instead of (M) and (m). We shall often deal with the following natural generalizations.

1. Let $1 \leqslant p \leqslant \infty$. Let μ be a non-trivial measure defined on a sigma-field Σ of subsets of a set S. For any μ-measurable scalar-valued function f defined on S, we let

$$\|f\|_p = \Big(\int_S |f(s)|^p \mu(ds)\Big)^{1/p} \quad \text{for} \quad 1 \leqslant p < \infty;$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{s\in S} |f(s)|.$$

(*) Numbers in brackets refer to the "Bibliography" as well as to the "Additional bibliography".

$L^p(\mu)$ is the Banach space (under the norm $\|\cdot\|_p$) of all classes of almost everywhere equal functions f defined on S such that $\|f\|_p < \infty$.

If S is an arbitrary non-empty set and μ is the measure defined for all subsets A of S by letting $\mu(A) = \infty$ if A is infinite and $\mu(A) =$ the cardinality of A otherwise, then the resulting space $L^p(\mu)$ will be denoted by $l^p(S)$.

In the case where S is finite and has n elements, the space $l^p(S)$ will be denoted by l_n^p.

2. By $c_0(S)$ we denote the closed linear subspace of $l^\infty(S)$ consisting of points $f \in l^\infty(S)$ such that, for every $\delta > 0$, the set $\{s \in S: |f(s)| > \delta\}$ is finite.

3. By $C(K)$ we denote the Banach space of all continuous scalar-valued functions defined on a compact Hausdorff space K, with the norm $\|f\| = \sup_{k \in K} |f(k)|$.

We shall be concerned with the Banach spaces over the fields of both real and complex scalars.

By a *subspace* of a Banach space X we shall aways mean a closed linear subspace of X.

For any Banach space X, we denote by X^* and X^{**} the dual (conjugate) and the second dual (second conjugate) of X. If $T: X \to Y$ is a continuous linear operator, then T^* and T^{**} denote the conjugate and the second conjugate operator of T.

In the sequel we shall use the phrases "linear operator", "continuous linear operator" and "bounded linear operator" as synonyms; the same concerns "linear functionals", etc.

The phrase "opération linéaire totalement continue" is translated as "compact linear operator".

By a *projection* on a Banach space X we shall mean a bounded linear projection, i.e. a bounded linear operator $P: X \to X$ which is an idempotent. A subspace of X which is a range of the projection is said to be *complemented in X*.

CHAPTER I

§ 1. Reflexive and weakly compactly generated Banach spaces Related counter-examples

Théorème 13 in [B], Chap. XI, was a starting point for many investigations. In order to state the results let us recall several, already standard, definitions.

The *weak topology* of a Banach space X is the weakest topology in which all bounded linear functionals on X are continuous. A subset $W \subset X$ is said to be *weakly compact* if it is compact in the weak topology of X; W is said to be *sequentially weakly compact* if, for every sequence of elements of W, there is a subsequence which is weakly convergent to an element of W. The map $\varkappa\colon X \to X^{**}$ defined by $(\varkappa x)(x^*) = x^*(x)$ for $x \in X$, $x^* \in X^*$ is called the *canonical embedding* of X into X^{**}. A Banach space X is said to be *reflexive* if $\varkappa(X) = X^{**}$. Banach's Théorème 13, which we mentioned at the beginning, characterizes reflexive spaces in the class of separable Banach spaces. The assumption of separability turns out to be superfluous. This is a consequence of the following fundamental fact, discovered by Eberlein [1] and Šmulian [1].

1.1. *A subset W of a Banach space X is weakly compact if and only if it is sequentially weakly compact.*

A simple proof of 1.1 was given by Whitley [1]. For other proofs and generalizations see Bourbaki [1], Köthe [1], Grothendieck [1], Ptak [1], Pełczyński [1].

From 1.1 we obtain the classical characterization of reflexivity generalizing Théorème 13 in [B], Chap. XI.

1.2. *For every Banach space X the following statements are equivalent:*

(i) *X is reflexive.*

(ii) *The unit ball of X is weakly compact.*

(iii) *The unit ball of X is sequentially weakly compact.*

(iv) *Every separable subspace of X is reflexive.*

(v) *Every descending sequence of bounded nonempty convex closed sets has a nonempty intersection.*

(vi) X^* *is reflexive.*

Many interesting characterizations of reflexivity have been given by James [4], [5]. One of them, James [3], is theorem 1.3 below (see James [6] for a simple proof). For simplicity we shall state this theorem only for real spaces.

1.3. *A real Banach space X is reflexive if and only if every bounded linear functional on X attains its maximum on the unit ball of X.*

It is interesting to compare 1.3 with the following theorem of Bishop and Phelps [1] (see also Bishop and Phelps [2]).

1.4. *For every real Banach space X, the set of bounded linear functionals which attain their least upper bounds on the unit ball is norm-dense in X^*.*

The reader interested in other characterizations of reflexivity is referred to Day [1], to the survey by Milman [1], to Köthe [1] and the references therein.

James supplied counter-examples showing that the assumptions of Théorème 13 in [B], Rem. XI, in general cannot be weakened and answering questions stated in [B], Rem. XI, § 9.

Example 1 (James [2]). Let J be the space of real or complex sequences $x = (x(j))_{1\leqslant j<\infty}$ such that $\lim_j x(j) = 0$ and

$$\|x\| = \sup\big(|x(p_1)-x(p_2)|^2+\ldots+|x(p_{n-1})-x(p_n)|^2+|x(p_n)-x(p_1)|^2\big)^{1/2} < \infty,$$

where the supremum is extended over all finite increasing sequences of indices $p_1 < p_2 < \ldots < p_n$ $(n = 1, 2, \ldots)$.

It is easily seen that J under the norm $\|\cdot\|$ is a separable Banach space.

1.5. *The space J has the following properties:*

(a) *J is isometrically isomorphic to J^{**}.*

(b) *$\varkappa(J)$ has codimension 1 in J^{**}, i.e.* $\dim J^{**}/\varkappa(J) = 1$.

(c) *There is no Banach space X over the field of complex numbers which, regarded as a real space, is isomorphic to the space J of real sequences* (Dieudonné [1]).

(d) *The space $J\times J$ is not isomorphic to any subspace of J* (Bessaga and Pełczyński [1]).

(e) *J is not weakly complete but has no subspace isomorphic to c_0.*

Statement (d) answers a question in [B], Rem. p. 214. Other examples of Banach spaces non-isomorphic to their Cartesian squares have been constructed by Semadeni [1] (cf. 11.20 in this article) and by Figiel [1].

Figiel's space is reflexive, while the dual of Semadeni's space is isomorphic to its Cartesian square.

In connection with question 1° in [B], Rem. XII, p. 215, we shall mention that all subspaces of codimension one (i.e. kernels of continuous linear functionals) of a given Banach space are isomorphic to each other but it is not known whether there exists an infinite-dimensional Banach space which is not isomorphic to its subspace of codimension one. However, there exist infinite-dimensional normed linear spaces (Rolewicz [1] and Dubinsky [1]) and infinite-dimensional locally convex complete linear metric spaces (Bessaga, Pełczyński and Rolewicz [1]) with this property.

Now we shall discuss another example of James [8].

EXAMPLE 2. Let $I = \{(n, i): n = 0, 1, 2, \ldots; 0 \leqslant i < 2^n\}$. Call a *segment* any subset of I of the form $(n, i_1), (n+1, i_2), \ldots, (n+m, i_m)$ such that $0 \leqslant i_{k+1} - 2i_k \leqslant 1$ for $k = 1, 2, \ldots, m-1$ $(n, m = 0, 1, \ldots)$. Let F denote the space of scalar-valued functions on I with finite supports. The norm on F is defined by the formula

$$\|x\| = \sup \Big(\sum_{q=1}^{p} \Big| \sum_{(n,i)\in S_q} x(n, i) \Big|^2 \Big)^{1/2},$$

with the supremum taken over all finite systems of pair-wise disjoint segments $S_1, S_2, \ldots, S_p$. The completion of F in the norm $\|\cdot\|$ will be denoted by DJ.

1.6. *The Banach space DJ has the following properties* (James [8]):

(a) *DJ is separable and has a non-separable dual.*

(b) *The unit ball of DJ is conditionally weakly compact, i.e. every bounded sequence* (x_n) *of elements of DJ contains a subsequence* (x_{k_n}) *such that* $\lim_n x^*(x_{k_n})$ *exists for every* $x^* \in (DJ)^*$.

(c) *Every separable infinite-dimensional subspace E of the space* $(DJ)^*$ *contains a subspace isomorphic to the Hilbert space* l^2.

(d) *No subspace of DJ is isomorphic to* l^1.

(e) *If B is the closed linear subspace of* $(DJ)^*$ *spanned by the functionals* f_{ni} *for* $0 \leqslant i < 2^n$; $n = 0, 1, \ldots$, *where* $f_{ni}(x) = x(n, i)$ *for* $x \in DJ$, *then* $B^* = DJ$ *and the quotient space* $(DJ)^*/B$ *is isomorphic to a non-separable Hilbert space* (Lindenstrauss and Stegall [1]).

Property (b) of the space J and property (e) of DJ suggest the following problem: Given a Banach space X, does there exist a Banach space Y such that the quotient space $Y^{**}/\varkappa(Y)$ is isomorphic to X? This problem is examined in the papers by James [7], Lindenstrauss [5], Davis, Figiel, Johnson and Pełczyński [1]. The results already obtained in this respect concern an important class of WCG Banach spaces.

A Banach space X is said to be WCG (an abbreviation for *weakly* *compactly generated*) if there exists a continuous linear operator from a reflexive Banach space to X whose range is dense in X (cf. Amir and Lindenstrauss [1], Davis, Figiel, Johnson and Pełczyński [1]). Obviously, every reflexive Banach space is WCG; also, it is easy to show that every separable space is WCG. We know that (Davis, Figiel, Johnson and Pełczyński [1]).

1.7. *For every* WCG *Banach space X there exists a Banach space Y such that the quotient space $Y^{**}/\varkappa(Y)$ is isomorphic to X.*

Setting $Z = Y^*$, we obtain

1.8. *If X is a* WCG *Banach space, then there exists a bounded linear operator Z: $Z^* \xrightarrow[\text{onto}]{} X$ such that Z^{**} is a direct sum of $\varkappa(Z)$ and the subspace $T^*(X)$ which is isometrically isomorphic to X^*.*

Moreover, if X is separable, then the space Z above can be so constructed that Z^* is separable and has a Schauder basis (Lindenstrauss [5]).

The WCG spaces have been introduced by Amir and Lindenstrauss [1]. They share many properties of finite-dimensional Banach spaces. Amir and Lindenstrauss [1] proved the following:

1.9. *If X is a* WCG *Banach space, then for every separable subspace E of X there exists a projection P: $X \to X$ of norm 1 whose range $P(X)$ contains E and is separable.*

The last result is a starting point for several theorems on renorming WCG spaces. Recall that, if E is a normed linear space with the original norm $\|\cdot\|$, then a norm p: $E \to R$ is equivalent to $\|\cdot\|$ if there is a constant $a > 0$ such that $a^{-1}p(x) \leqslant \|x\| \leqslant ap(x)$ for $x \in X$. Troyansky [1] has proved the following:

1.10. *For every* WCG *Banach space X there exists an equivalent norm p which is locally uniformly convex, i.e. for every $x \in X$ with $p(x) = 1$ and for every sequence (x_n) in X, the condition $\lim_n p(x_n) = 2^{-1} \lim_n p(x+x_n) = 1$ implies $\lim_n p(x-x_n) = 0$.*

In particular, the norm p is strictly convex, i.e. $p(x)+p(y) = p(x+y)$ implies the linear dependence of x and y.

Assertion 1.10 for separable Banach spaces is due to Kadec [1], [2]. The existence of an equivalent strictly convex norm for WCG spaces has been established by Amir and Lindenstrauss [1].

In connection with 1.10 let us mention the following result of Day [2]:

1.11. *The space $l^\infty(S)$ with uncountable S admits no equivalent strictly convex norm.*

More information on renorming theorems can be found in Day [1] and papers by Asplund [1], [2], Lindenstrauss [6], Troyansky [1], Davis and Johnson [1], Klee [1], Kadec [2], Kadec and Pełczyński [2], Whitfield [1], Restrepo [1].

In contrast to the case of separable and reflexive Banach spaces we have (Rosenthal [1])

1.12. *There exists a Banach space X which is not* WCG *but is isomorphic to a subspace of a* WCG *space.*

Concluding this section, we shall discuss one more example.

EXAMPLE 3 (Johnson and Lindenstrauss [1]). Let S be an infinite family of subsets of the set of positive integers which have finite pair-wise intersections (cf. Sierpiński [1]). Let E_0 be the smallest linear variety in l^∞ containing all characteristic functions χ_A for $A \in S$ and all sequences tending to zero. It is easily seen that the formula

$$|||y||| = \Big\|x + \sum_{j=1}^{n} c_{A_j}\chi_{A_j}\Big\|_\infty + \Big(\sum_{j=1}^{n} |c_{A_j}|^2\Big)^{1/2} \quad \text{for} \quad y = \sum_{j=1}^{n} c_{A_j}\chi_{A_j},$$

where $x \in c_0$ and $A_1, \ldots, A_n \in S$ $(n = 1, 2, \ldots)$, defines a norm on E_0. The coefficient functionals $g_k(y) = y(k)$ for $y \in E_0$ are continuous in this norm. Let E be the Banach space which is the completion of E_0 in the norm $|||\cdot|||$ and let f_k be the continuous linear functional on E which extends g_k $(k = 1, 2, \ldots)$. Then

1.13. *The space E has the following properties:*

(a) *The linear functionals $f_1, f_2, \ldots$ separate points of E.*

(b) *E is not isomorphic to a subspace of any* WCG *space, in particular E is not isomorphically embeddable into l^∞.*

(c) *E^* is isomorphic to the product $l^1 \times l^2(S)$, hence it is* WCG.

CHAPTER II

Local properties of Banach spaces

§ 2. The Banach–Mazur distance and projection constants

The distance between isomorphic Banach spaces introduced in [B], Rem. XI, § 6, p. 212, plays an important role in the recent investigations of isomorphic properties of Banach spaces, and in particular in the study of the properties of finite-dimensional subspaces of a given Banach space X, which are customarily referred to as the "local properties" of the space X.

Let $a \geqslant 1$. Banach spaces X and Y are said to be *a-isomorphic* if there exists an isomorphism T of X onto Y such that $\|T\| \cdot \|T^{-1}\| \leqslant a$. The infimum of the numbers a for which X and Y are a-isomorphic is called the *Banach–Mazur distance* between X and Y and is denoted by $d(X, Y)$. Obviously 1-isomorphisms are the same as isometrical isomorphisms.

2.1. *There exist Banach spaces X_0, X_1 with $d(X_0, X_1) = 1$ which are not isometrically isomorphic.*

Proof. Consider in the space c_0 two norms

$$\|x\|_i = \sup_j |x(j)| + \Big(\sum_{j=1}^{\infty} |2^{-j} x(j+i)|^2\Big)^{1/2} \quad \text{for} \quad x = (x(j));\ i = 0, 1.$$

For $i = 0, 1$, let X_i be the space c_0 equipped with the norm $\|\cdot\|_i$. For $n = 1, 2, \ldots$, let $T_n\colon X_0 \to X_1$ be the map defined by

$$(x(1), x(2), \ldots) \to (x(n), x(1), \ldots, x(n-1), x(n+1), \ldots).$$

Then each T_n is an isomorphism of X_0 onto X_1 and $\lim_n \|T_n\|\,\|T_n^{-1}\| = 1$. Hence $d(X_0, X_1) = 1$. On the other hand, the norm $\|\cdot\|_0$ is strictly convex (for the definition see section 1 after 1.10) while $\|\cdot\|_1$ is not. Therefore X_0 is not isometrically isomorphic to X_1.

Let us mention that $d(c, c_0) = 3$, which is related to a question in [B], Rem. XI, § 6, pp. 212–213. Interesting generalizations of this fact are due to Cambern [1] and Gordon [1]; see also 10.19 and the comment after it.

From the compactness argument it follows that, for arbitrary Banach spaces X, Y of the same finite dimension, there exists a $d(X, Y)$-isomorphism of X onto Y.

The following important estimation is due to John [1]:

2.2. *If X is an n-dimensional Banach space, then $d(X, l_n^2) \leqslant \sqrt{n}$.*

Since $d(l_n^\infty, l_n^2) = \sqrt{n}$ (cf. 2.3), the estimation above is the best possible. The exact rate of growth of the sequence (d_n), where $d_n = \sup\{d(X, Y)$: $\dim X = \dim Y = n\}$, is unknown. From 2.2 and the "triangle inequality" $d(X, Z) \leqslant d(X, Y) \cdot d(Y, Z)$ it follows that $\sqrt{n} \leqslant d_n \leqslant n$ for $n = 1, 2, \ldots$

The computation of the Banach–Mazur distance between given isomorphic Banach spaces is rather difficult. Gurariĭ, Kadec and Macaev [1], [2] have found that

2.3. *If either $1 \leqslant p < q \leqslant 2$ or $2 \leqslant p < q \leqslant \infty$, then*

$$d(l_n^p, l_n^q) = n^{1/p-1/q} \qquad (n = 1, 2, \ldots);$$

if $1 \leqslant p < 2 < q \leqslant \infty$, then

$$(\sqrt{2}-1)\, d(l_n^p, l_n^q) \leqslant \max(n^{1/p-1/2}, n^{1/2-1/q}) \leqslant \sqrt{2}\, d(l_n^p, l_n^q) \qquad (n = 1, 2, \ldots).$$

For generalizations of 2.3 to the case of spaces with symmetric bases and some matrix spaces see Gurariĭ, Kadec and Macaev [2], [3], Garling and Gordon [1].

Estimations of the Banach–Mazur distance are related to the computation of so called "projection constants". Let $a \geqslant 1$ and let X be a Banach space. A subspace Y of X is *a-complemented* in X if there exists a linear projection $P\colon X \underset{\text{onto}}{\longrightarrow} Y$ with $\|P\| \leqslant a$. The infimum of the numbres a such that Y is a-complemented in X will be denoted $p(Y, X)$. For any Banach space E we let

$$p(E) = \sup p\big(i(E), X\big),$$

where the supremum is extended over all Banach spaces X and all isometrically isomorphic embeddings $i\colon E \to X$. The number $p(X)$ is called the *projection constant* of the Banach space E.

In general, if $\dim E = \infty$, then $p(E) = \infty$. No characterization of the class of Banach spaces E with $p(E) < \infty$ is known (cf. section 11). The projection constant of a Banach space E is closely related to extending linear operators with values in E.

2.4. *Let E be a Banach space. If $p(E) < \infty$, then, for every triple (X, Y, T) consisting of a Banach space X, its subspace Y and a continuous linear operator $T\colon Y \to E$ and for every $\varepsilon > 0$, there exists a linear operator $\tilde{T}\colon X \to E$ such that*

$$(*) \qquad \tilde{T} \text{ extends } T \quad \text{and} \quad \|\tilde{T}\| \leqslant C \cdot \|T\|$$

with $C = p(E)+\varepsilon$. *Conversely, if for every triple* (X, Y, T) *there is a* T *satisfying* $(*)$, *then* $p(E) \leqslant C$. *We have* $p(E) = \infty$ *if and only if there exists a triple* (X, Y, T) *such that* T *admits no extension to a bounded linear operator defined on the whole of* X.

Using the theorem of John 2.2, Kadec and Snobar [1] have shown that

2.5. *If* $\dim X = n$, *then* $p(X) \leqslant \sqrt{n}$ $(n = 1, 2, \ldots)$.

The estimation 2.5 gives the best rate of growth. We find that (Grünbaum [1], Rutowitz [1], Daugavet [1])

2.6. $p(l_n^2) = \pi^{-1/2} n\Gamma\left(\frac{n}{2}\right)/\Gamma\left(\frac{n+1}{2}\right) \sim \sqrt{2n/\pi}$ $(n = 2, 3, \ldots)$.

Rutovitz [1] and Garling and Gordon [1] estimated projection constants of the spaces l_n^p.

2.7. *If* $2 \leqslant p \leqslant \infty$, *then* $p(l_n^p) = n^{1/p}\alpha_p(n)$, *where* $1/\sqrt{2} < \alpha_p(n) \leqslant \alpha_\infty(n) = 1$ $(n = 1, 2, \ldots)$. *If* $1 \leqslant p \leqslant 2$, *then* $p(l_n^p) = n^{1/2}\alpha_p(n)$, *where* $1 \geqslant \alpha_p(n) \geqslant \left(\sinh\frac{\pi}{2}\right)^{-1}$ $(n = 1, 2, \ldots)$.

Remark. Theorem 2.7 concerns real spaces l_n^p, however, in the complex case the rate of growth is the same.

For generalizations of 2.7 to spaces with symmetric bases see Garling and Gordon [1] and the references therein.

By 2.7 we have in particular $p(l_n^\infty) = 1$ for $n = 1, 2, \ldots$; the last property isometrically characterizes the spaces l_n^∞ in the class of finite-dimensional Banach spaces (see Nachbin [1] and 10.15).

It is easy to show that $p(X) \leqslant d(X, l_n^\infty)$ for every n-dimensional Banach space X. It is not known whether the quantities $p(X)$ and $d(X, l_n^\infty)$ are of the same rate of growth, i.e. whether there exists a constant $K > 0$ independent of n and such that $d(X, l_n^\infty) \leqslant Kp(X)$ for every n-dimensional Banach space X. Also, the numbers

$$c_n = \sup\{p(X)\colon \dim X = n\} \quad \text{for} \quad n = 2, 3, \ldots$$

have not been computed. Some results concerning the last problem are given in Gordon [2].

The Banach–Mazur distance and projection constants are connected with other isometric invariants of finite-dimensional Banach spaces. The asymptotic behaviour of these invariants in some classes of finite-dimensional Banach spaces with the dimensions growing to infinity gives rise to isomorphic invariants of infinite-dimensional Banach spaces. These problems have many points in common with the theory of Banach ideals. The interested reader is referred to Grothendieck [5], [6], Lindenstrauss and Pełczyński [1], Pietsch [1] with references, Gordon [2], [3], [4], Garling and Gordon [1], Gordon and Lewis [1], Gordon, Lewis and Retherford [1], [2], Snobar [1], Pietsch [1], Milman and Wolfson [1], Figiel, Lindenstrauss and Milman [1].

§ 3. Local representability of Banach spaces

The following concept, introduced by Grothendieck [6] and James [10], originates from the Banach–Mazur distance.

Let $a \geqslant 1$. A Banach space X is *locally a-representable* in a Banach space Y, if for every $b > a$ every finite-dimensional subspace of X is b-isomorphic to a subspace of Y. If X is locally a-representable in Y and Y is locally a-representable in X, we say that X is *locally a-isomorphic* to Y. The space X is said to be *locally representable* in Y (*locally isometric* to Y) if X is locally 1-representable in Y (locally 1-isomorphic to Y).

First, we shall discuss the problem of finding Banach spaces with are locally representable in the spaces l^p $(1 \leqslant p < \infty)$ and c_0. We know (Grothendieck [5], Joichi [1], cf. also 9.7) that

3.1. *A Banach space X is locally a-representable in l^2 if and only if X is a-isomorphic to l^2.*

Theorem 3.1 can be generalized to the case of l^p with $1 \leqslant p < \infty$ (Bretagnolle, Dacunha-Castelle and Krivine [1], Bretagnolle and Dacunha-Castelle [1], Dacunha-Castelle and Krivine [1], Lindenstrauss and Pełczyński [1]) as follows:

3.2. *Let $1 \leqslant p < \infty$ and let $a \geqslant 1$. A Banach space X is locally a-representable in l^p if and only if X is a-isomorphic to a subspace of a space $L^p(\mu)$ (in particular to a subspace of L^p when X is separable).*

Thus, by the results of Schoenberg [1], [2], the local representability of a Banach space X in some l^p for $1 \leqslant p \leqslant 2$ can be characterized by the fact that the norm of X is negative definite. For $2n < p \leqslant 2n+2$ $(n = 1, 2, \ldots)$ more sophisticated conditions have been found by Krivine [1].

The last theorem is also valid for $p = \infty$. In fact, we have

3.3. (i) *For every cardinal $n \geqslant \aleph_0$, there is a compact Hausdorff space K such that the topological weight of the space $C(K)$ is n and every Banach space whose topological weight is $\leqslant n$ is isometrically isomorphic to a subspace of the space $C(K)$.*

(ii) *Every Banach space is locally representable in the space c_0.*

Statement (i) generalizes the classical Banach–Mazur theorem ([B], Chap. XI, Théorème 9), which says that every separable Banach space is isometrically isomorphic to a subspace of C. The proof of (i) is almost the same as that of Théorème 9 but, instead of using the fact that every compact metric space is a continuous image of the Cantor set, it employs the theorem of Esenin-Volpin [1] (which was proved under the continuum hypothesis), stating that for every cardinal $n \geq \aleph_0$ there is a compact Hausdorff space K of the topological weight n such that every compact Hausdorff space of topological weight $\leq n$ is a continuous image of K.

Statement (ii) follows from the fact that every centrally symmetric k-dimensional polyhedron with $2n$ vertices is affinely equivalent to the intersection of the cube $[-1,1]^n$ (the unit ball of the space l_n^∞) with a k-dimensional subspace of l_n^∞ for $k = 1, 2, \ldots$; $n \geq k$ (Klee [2]).

Next consider the problem: Given $p \in [1, \infty]$, characterize Banach spaces in which l^p is locally representable. We present answers for $p = 1, 2, \infty$. (The case of arbitrary p, due to Krivine [2] (cf. also Maurey and Pisier [3], Rosenthal [9]) is much more difficult.) The following beautiful result is due to Dvoretzky [1]:

3.4. *The space l^2 is locally representable in every infinite-dimensional Banach space.*

This result is a simple consequence of the following fact concerning convex bodies:

3.5 (*Dvoretzky's theorem on almost spherical sections*). *For every $\varepsilon > 0$ and for every positive integer k, there exists a positive integer $N = N(k, \varepsilon)$ such that every bounded convex body (= convex set with non-empty interior) B in the real or complex space l_N^2 which is symmetric with respect to the origin admits an intersection with a k-dimensional subspace Y which approximates up to ε a Euclidean k-ball, i.e.*

$$\sup\{\|x\|:\ x \in Y \cap K\}/\inf\{\|x\|:\ x \in Y \backslash K\} < 1+\varepsilon.$$

The proof of the real version of 3.5 is due to Dvoretzky [2] (previously it was announced in Dvoretzky [1]). Some completions and simplifications can be found in Figiel [2]. An essentially simpler proof, based on a certain isoperimetric theorem of P. Levy, has been given by Milman [2], cf. also Figiel, Lindenstrauss and Milman [1]. The proof of Figiel [5] basing on an idea of Szankowski [1] is short and elegant.

Banach spaces with unconditional bases (for the definition see § 7) have the following property (Tzafriri [1]):

3.6. *If X is an infinite-dimensional Banach space with an unconditional basis, then there exsist a constant M, a sequence of projections P_n: $X \to X$ with $\|P_n\| \leq M$ for $n = 1, 2, \ldots$ and a $p \in \{1, 2, \infty\}$ such that $\sup_n d(P_n(X), l_n^p) \leq M$.*

The proof of 3.6 is based on the Brunel–Sucheston [1] technique of constructing sub-symmetric bases, which employs a certain combinatorial theorem of Ramsey [1]. A similar argument yields also the following weaker version of Dvoretzky's theorem: *For every infinite-dimensional Banach space X there is an $a \geqslant 1$ such that l^2 is a-representable in X.*

Characterizations of Banach spaces in which c_0, equivalently l^∞, is locally represented are connected with the theory of random series. Recall that a measurable real function f on a probabilistic space (Ω, μ) is called a *standard Gaussian random variable* if

$$\mu\{\omega \in \Omega : f(\omega) < t\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2}\, ds.$$

The Rademacher functions $(r_j)_{1 \leqslant j < \infty}$ are defined on the interval $[0, 1]$ by the formula

$$r_j(t) = \operatorname{sgn} \sin 2^j \pi t, \quad j = 1, 2, \ldots$$

We have

3.7. *For every Banach space X the following statements are equivalent:*

(i) *The space c_0 is not locally a-representable in X for any $a \geqslant 1$.*

(ii) *The space c_0 is not locally representable in X.*

(iii) *The space c_0 is not locally representable in the product space $(X \times X \times \ldots)_{l^p}$ for any $p \in [1, \infty)$.*

(iv) *There are a $q \in [2, \infty)$ and a constant $C > 0$ such that*

$$\Big(\sum_{j=1}^{n} \|x_j\|^q\Big)^{1/q} \leqslant C \int_0^1 \Big\|\sum_{j=1}^{n} r_j(t) x_j\Big\| \, dt$$

for arbitrary $x_1, \ldots, x_n \in X$ and $n = 1, 2, \ldots$

(v) *For every sequence (x_n) of elements of X and for every sequence of independent standard Gaussian random variables, the series $\sum_n f_n(\omega) x_n$ converges almost everywhere iff so does the series $\sum_n r_n(t) x_n$.*

The equivalence between (i) and (ii) has been proved by Giesy [1]. The other implications in 3.7 are due to Maurey and Pisier [2]. Other equivalent conditions, stated in terms of factorizations of compact linear operators, can be found in Figiel [3].

The next theorem characterizes Banach spaces in which the space l_1 is not locally representable.

3.8. *For every Banach space X the following statements are equivalent:*

(i) *The space l^1 is not locally a-representable in X for any $a \geqslant 1$.*

(ii) *The space l^1 is not locally representable in X.*

(iii) *The space l^1 is not locally representable in the product space $(X \times X \times \ldots)_{l^p}$ for any $p \in (1, \infty)$.*

(iv) *There are a* $q \in (1, \infty)$ *and a constant* $C > 0$ *such that*

$$\int_0^1 \Big\| \sum_{i=1}^n r_i(t) x_i \Big\| dt \leqslant C \Big(\sum_{i=1}^n \|x_i\|^q \Big)^{1/q}$$

for arbitrary $x_1, \ldots, x_n \in X$ *and* $n = 1, 2, \ldots$

(v) *There are a* $q \in (1, \infty)$ *and a constant* $C > 0$ *such that*

$$\operatorname*{ess\,inf}_{0 \leqslant t \leqslant 1} \Big\| \sum_{i=1}^n r_i(t) x_i \Big\| \leqslant C \Big(\sum_{i=1}^n \|x_i\|^q \Big)^{1/q}$$

for arbitrary $x_1, \ldots, x_n \in X$ *and* $n = 1, 2, \ldots$

The equivalence between (i) and (ii) has been proved by Giesy [1]. The other implications in 3.8 are due to Pisier [1].

Let us notice in connection with 3.7 and 3.8 that if a Banach space X has a subspace isomorphic either to l^1 or to c_0, then, for every $a \geqslant 1$, there is a subspace of X which is a-isomorphic to l^1 or c_0, respectively (James [9]). It is not known whether the spaces l^p with $1 < p < \infty$ have an analogous property.

Obviously, if a Banach space X has a subspace isometrically isomorphic to a space l^p or c_0, then the space l^p or c_0, respectively, is locally a-representable in X for some $a \geqslant 1$. Converse implications are, in general, false. The spaces l^p for $1 \leqslant p < \infty$, $p \neq 2$, and c_0 do not contain any subspace isomorphic to l^2 (cf. 12.) in contrast to Dvoretzky's theorem 3.5. Even more "pathological" in this respect is the example due to Tzirelson [1]. Below we present a modified version of this example given by Figiel and Johnson [2].

EXAMPLE. Let E_0 be the space of all scalar sequences having at most finitely many non-zero coordinates and let $(\|\cdot\|_n)$ be the sequence of norms on E_0 defined by

$$\|x\|_0 = \sup_k |x(k)|,$$

$$\|x\|_{n+1} = \max\Big(\|x\|_n, \tfrac{1}{2} \sum_{j=1}^m \Big\| \sum_{i=\nu(j-1)+1}^{\nu(j)} x(i) e_i \Big\|_n\Big),$$

where $e_i = (0, 0, \ldots, \underset{\text{the } i\text{th place}}{1}, 0, \ldots)$, and the supremum is extended over all increasing finite sequences of indices $\nu(0) < \nu(1) < \ldots < \nu(m)$ such that $\nu(0) \geqslant m$. Let

$$\|x\| = \lim_n \|x\|_n \quad \text{for} \quad x \in E_0.$$

It is easy to show that the limit above exists. Let E be the completion of E_0 in the norm $\|\cdot\|$. Then

3.9. *E is a separable Banach space with an unconditional basis which does not contain isomorphically any space l^p $(1 \leqslant p \leqslant \infty)$ or c_0.*

Concluding this section, we shall state a theorem of general nature indicating the difference between the local and the global structure of Banach spaces.

3.10 (*The Principle of Local Reflexivity*). *Every Banach space is locally isometric to its second dual.*

This fact is a consequence of the following result. (For simplicity we identify the Banach space X with its canonical image $\varkappa(X)$ in X^{**}.)

3.11. *Let X be a Banach space, let E and G be finite-dimensional subspaces of X^{**} and X^*, respectively, and let $0 < \varepsilon < 1$. Assume that there is a projection P of X^{**} onto E with $\|P\| \leqslant M$. Then there are a continuous linear operator $T\colon E \to X$ and a projection P_0 of X onto $T(E)$ such that*

(a) $T(e) = e$ *for* $e \in E \cap X$.

(b) $f(Te) = e(f)$ *for* $e \in E$ *and* $f \in G$.

(c) $\|T\| \cdot \|T^{-1}\| \leqslant 1+\varepsilon$.

(d) $\|P_0\| \leqslant M(1+\varepsilon)$.

Moreover, if $P = Q^*$ where Q is a projection of X^* into X^*, then the projection P_0 can be chosen so as to satisfy (*d*) *and the additional condition*

(e) $P_0^{**}(x^{**}) = P(x^{**})$ *whenever* $P(x^{**}) \in X$.

Theorem 3.10. and a part of 3.11 have been given by Lindenstrauss and Rosenthal [1]. Theorem 3.11 in the present formulation is due to Johnson, Rosenthal and Zippin [1]. For an alternative proof see Dean [1].

§ 4. The moduli of convexity and smoothness; super-reflexive Banach spaces Unconditionally convergent series

Intensive research efforts have been devoted to the invariants of the local structure of Banach spaces related to the geometrical properties of their unit spheres. In this section we shall discuss two invariants of this type: the modulus of convexity (Clarkson [1]) and the modulus of smoothness (Day [3]).

Let X be a Banach space; for $t > 0$, we set

$$\delta_X(t) = \inf\{1-\tfrac{1}{2}\|x+y\|\colon \|x\| = \|y\| = 1,\ \|x-y\| \geqslant t\},$$

$$\varrho_X(t) = \tfrac{1}{2}\sup\{\|x+y\|+\|x-y\|-2\colon \|x\| = 1,\ \|y\| = t\}.$$

The functions δ_X and ϱ_X are called, respectively, the *modulus of convexity* and the *modulus of smoothness* of the Banach space X. The space X is said to be *uniformly convex* (resp. *uniformly smooth*) if $\delta_X(t) > 0$ for $t > 0$ (resp. $\lim_{t\to 0} \varrho_X(t)/t = 0$).

The moduli of convexity and smoothness are in a sense dual to each other. We have (Lindenstrauss [8], cf. also Figiel [6]).

4.1. *For every Banach space X, $\varrho_{X^*}(t) = \sup_{0\leqslant s\leqslant 2}\big(ts/2-\delta_X(s)\big)$.*

The next result characterizes the class of Banach spaces for which one can define an equivalent uniformly convex (smooth) norm.

4.2. *For every Banach space X the following conditions are equivalent:*

(a) *X is isomorphic to a Banach space which is both uniformly convex and uniformly smooth.*

(b) *X is isomorphic to a uniformly smooth space.*

(c) *X is isomorphic to a uniformly convex space.*

(d) *Every Banach space which is locally a-representable in X, for some $a > 1$, is reflexive.*

(e) *Every Banach space locally representable in X is reflexive.*

(f) *The dual space X^* satisfies conditions* (a)–(e).

A Banach space satisfying the equivalent conditions of 4.2 is said to be *super-reflexive.*

Theorem 4.2 is a product of combined efforts of R. C. James [10], [11] and Enflo [2]. The implication: "(b) and (c) ⇒ (a)" has been proved by Asplund [2]. For the characterizations of super-reflexivity in terms of "geodesics" on the unit spheres see James and Schaffer [1], and in terms of basic sequences, see V. I. Gurariĭ and N. I. Gurariĭ [1] and James [12].

If X is a super-reflexive Banach space, then by (e) neither l^1 nor c_0 is locally representable in X. Therefore the product

$$(l_1^1 \times l_2^1 \times l_3^1 \times \ldots)_{l^2}$$

is an example of a reflexive Banach space which is not super-reflexive. A much more sophisticated example is due to James [13], who proved that

4.3. *There exists a reflexive Banach space RJ which is not super-reflexive but is such that l^1 is not locally representable in RJ.*

Clarkson [1] has shown that, for $1 < p < \infty$, the spaces L^p and l^p are uniformly convex. The exact values of $\delta_X(t)$ for $X = L^p, l^p$ have been computed by Hanner [1] and Kadec [5]. Their results together with 4.1 yield the following asymptotic formulas:

4.4. *If X is either L^p or l^p with $1 < p < \infty$, then*

$$\delta_X(t) = a_p t^k + o(t^k), \qquad \varrho_X(t) = b_p t^m + o(t^m),$$

with $k = \max(2, p)$, $m = \min(2, p)$, where a_p and b_p are suitable positive constants depending only on p. Moreover, if Y is a uniformly convex (resp. uniformly smooth) Banach space which is isomorphic to L^p or l^p, then, for small positive t, we have $\delta_Y(t) \leqslant \delta_{l^p}(t)$ (resp. $\varrho_Y(t) \geqslant \varrho_{l^p}(t)$).

Orlicz spaces (i.e. the spaces (*o*) and (*O*) in the terminology of [B], pp. 202–203) admit equivalent uniformly convex norms iff they are reflexive (see Milnes [1]).

The moduli of convexity and smoothness are connected with the properties of unconditionally convergent series in the space X. Let us notice that the property: "*the series* $\sum_n \epsilon_n x_n$ *of elements of a Banach space* X *is convergent for every sequence of signs* (ϵ_n)" is equivalent to the unconditional convergence of the series in the sense of Orlicz [3], cf. [B], Rem. IX, § 4.

We have

4.5. *If* $\sum_n \epsilon_n x_n$ *with* x_n*'s in a uniformly convex Banach space* X *is convergent for every sequence of signs* (ϵ_n), *then* $\sum_{n=1}^{\infty} \delta_X(\|x_n\|) < \infty$.

If $\sum_{n=1}^{\infty} \epsilon_n x_n$ *with* x_n*'s in a uniformly smooth Banach space* X *is divergent for every sequence of signs* (ϵ_n), *then* $\sum_{n=1}^{\infty} \varrho_X(\|x_n\|) = \infty$.

The first statement of 4.5 is due to Kadec [5], the second to Lindenstrauss [8].

Combining 4.4 with 4.5, we obtain (Orlicz [1], [2])

4.6. *Let* $1 < p < \infty$. *If* $\sum_n f_n$ *is an unconditionally convergent series in the space* L^p *(or more generally, in* $L^p(\mu)$*), then* $\sum_{n=1}^{\infty} \|f_n\|^{c(p)} < \infty$, *where* $c(p) = \max(p, 2)$.

The last fact is also valid for the space L^1, which is non-reflexive, and hence is not uniformly convex. We have (Orlicz [1])

4.7. *If in the space* L^1 *the series* $\sum_n f_n$ *is unconditionally convergent, then* $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$.

The exponents $c(p)$ in 4.6 and 2 in 4.7 are the best possible. This can easily be checked directly for $p > 2$; for $1 \leqslant p \leqslant 2$ it follows from the crucial theorem on unconditionally convergent series due to Dvoretzky and Rogers [1] (cf. also Figiel, Lindenstrauss and Milman [1]).

4.8. *Let* (a_n) *be a sequence of positive numbers such that* $\sum_{n=1}^{\infty} a_n^2 < \infty$. *Then in every infinite-dimensional Banach space* X *there exists an unconditionally convergent series* $\sum_n x_n$ *such that* $\|x_n\| = a_n$ *for* $n = 1, 2, \ldots$ *In particular, in every infinite-dimensional Banach space there is an unconditionally convergent series* $\sum_n x_n$ *such that* $\sum_{n=1}^{\infty} \|x_n\| = \infty$.

Combining 4.8 with 4.5, we get

4.9. *For every Banach space X there exist positive constants a and b such that $\delta_X(t) \leqslant at^2$ and $\varrho_X(t) \geqslant bt^2$ for small $t > 0$.*

Concluding our discussion, we shall state another theorem on unconditionally convergent series, which generalizes the theorem of Orlicz [1] (mentioned in [B], Rem. IX, § 4, p. 211).

4.10. *For every Banach space X the following statements are equivalent:*

(a) *For every series $\sum_n x_n$ of elements of X, if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$, then the series $\sum_n x_n$ is unconditionally convergent.*

(b) *For every series $\sum_n x_n$ of elements of X the condition*

$$\sup_n \Big\| \sum_{k=1}^{n} r_k(t) x_k \Big\| < \infty$$

almost everywhere on $[0, 1]$ implies the unconditional convergence of the series $\sum_n x_n$. (Here r_n denotes the n-th Rademacher function for $n = 1, 2, \ldots$).

(c) *No subspace of X is isomorphic to c_0.*

The equivalence of conditions (a) and (c) is proved in Bessaga and Pełczyński [3]. The equivalence of (b) and (c) is due to Kwapień [2].

There is ample literature concerning the moduli of convexity and smoothness and other related invariations of Banach spaces. In addition to the references already given in the text, the reader may consult books by Day [1], Chapt. VII, § 2, Lindenstrauss and Tzafriri [1], [2], the surveys by Milman [1], Zizler [1], Cudia [2], Lindenstrauss [4], [6] and papers by Asplund [1], Bonic and Frampton [1], Cudia [1], Day [4], Day, James and Swaminathan [1], Figiel [1], Figiel and Pisier [1], V. I. Gurariĭ [2], [3], [4], Henkin [1], Lovaglia [1], Nordlander [1], [2].

The theory of unconditionally convergent series is related to the theory of absolutely summing operators, originated by Grothendieck, and radonifying operators in the sense of L. Schwartz, which is a branch of measure theory in infinite-dimensional linear spaces. The interested reader is referred to the following books and papers: Grothendieck [1], [2], Pietsch [1], [2], [3], Persson and Pietsch [1], Lindenstrauss and Pełczyński [1], Maurey [1], Kwapień [1], L. Schwartz [1], [2].

For further information see "Added in proof".

CHAPTER III

The approximation property and bases

There are many instances in operator theory where it is convenient to represent a given linear operator as a limit of a sequence of operators with already known properties. The best investigated classes of operators are finite rank operators and compact operators, therefore it is natural to ask whether every continuous linear operator can be approximated by linear operators from these classes. Such a question was raised in [B], Rem. VI, § 1, p. 209. Banach, Mazur and Schauder have already observed that the approximation problem is related to the problem of existence of a basis, and to some questions on the approximation of continuous functions (cf. Scottish Book [1], problem 157). A detailed study by Grothendieck [4] published in the middle fifties explained the fundamental role of the approximation problem in the structure theory of Banach spaces, and that this problem arises in various contexts (for instance, if one attempts to determine the trace of a nuclear operator). Substantial progress was made in 1972 by Enflo [3], who constructed the first example of a Banach space which does not have the approximation property.

§ 5. The approximation property

We begin with some notation. By an *operator* we shall mean a continuous linear operator. For arbitrary Banach spaces X and Y, we denote

$\boldsymbol{B}(X, Y)$ = the space of all operators from X into Y,

$\boldsymbol{K}(X, Y)$ = the space of all compact operators from X into Y,

$\boldsymbol{F}(X, Y)$ = the space of all finite rank operators from X into Y.

For any $T \in \boldsymbol{B}(X, Y)$, we let $|||T||| = \sup\{\|Tx\|: \|x\| \leqslant 1\}$, the operator norm of T.

Definition. A Banach space Y *has the* ap (= *the approximation property*) if every compact operator with range in Y is the limit, in the operator norm, of a sequence of finite rank operators, i.e. for every Banach space X

and for every $K \in \boldsymbol{K}(X, Y)$, there exist $F_n \in \boldsymbol{F}(X, Y)$ $(n = 1, 2, \ldots)$ such that $\lim_n |||F_n - K||| = 0$.

The approximation property can easily be expressed in intrinsic terms of Y. We have (cf. Grothendieck [4] and Schaefer [1], Chap. III, § 9)

5.1. *For every Banach space Y the following statements are equivalent:*

(i) *Y has the* ap,

(ii) *given a compact subset C of Y, there exists a finite rank operator $F \in \boldsymbol{F}(Y, Y)$ such that $\|Fy - y\| < 1$ for all $y \in C$.*

The celebrated result of Enflo [3] on the existence of a Banach space which fails the ap has been improved by Davie [1], [2], Figiel [4] and Szankowski [2] as follows:

5.2. *For every $p \in [1, \infty]$, $p \neq 2$, there exists a subspace E_p of the space l^p which does not have the approximation property. Moreover, $E_\infty \subset c_0$.*

Davie's proof is short and elegant. It uses some properties of random series. Figiel's proof seems to be the most elementary. For other proofs of Enflo's theorem and related theorems see Figiel and Pełczyński [1] and Kwapień [4]. Kwapień's result seems to be interesting also from the point of view of harmonic analysis. He has shown that

5.3. *For each p with $2 < p < \infty$, there exist increasing sequences (n_k) and (m_k) of positive integers such that the closed linear subspace of L^p spanned by the functions $f_k(t) = e^{in_k 2\pi t} + e^{im_k 2\pi t}$ $(k = 1, 2, \ldots)$ fails the approximation property.*

It is interesting to compare 5.2 with the observation by W. B. Johnson [3] that there is a Banach space which is not isomorphic to a Hilbert space but such that every subspace of the space has ap.

Starting from one example of a Banach space which does not have the ap, one can construct further examples by passing to the dual space and taking products, because the approximation property is preserved under these operations. We have

5.4. *Any complemented subspace of a Banach space having the* ap *has the* ap.

5.5. *Let (E_i) be a sequence of Banach spaces each having the* ap. *Then the product $(E_1 \times E_2 \times \ldots)_{l^p}$ has the* ap *for $1 \leqslant p < \infty$.*

5.6 (Grothendieck [4]). *If X^* has the* ap, *then so does X.*

The last result is an easy consequence of the improved Local Reflexivity Principle 3.11.

It is interesting to note that the converse of 5.6 is false. Namely, from 1.8 it follows that

5.7 (Lindenstrauss [5]). *There exists a Banach space which has the* ap *(even has a basis) but whose dual does not have the* ap.

W. B. Johnson [1] gave a simple construction of such a space. Let (B_n) be a sequence of finite-dimensional Banach spaces such that, for every $\varepsilon > 0$ and for every finite-dimensional Banach space B, there exists an index n_0 such that $d(B, B_{n_0}) < 1+\varepsilon$. Let us set

$$BJ = (B_1 \times B_2 + \ldots)_{l^1}.$$

Then the space BJ has the following universality property:

5.8. *The conjugate of any separable Banach space is isomorphic to a complemented subspace of the space* $(BJ)^*$.

The space E_p of 5.2, being separable and reflexive for $1 < p < \infty$, is a conjugate of a separable Banach space. Hence, by 5.4 and 5.8, $(BJ)^*$ does not have the ap. On the other hand, it follows from 5.5 and the fact that every finite-dimensional Banach space has the ap that the space BJ has the approximation property.

The next two results do not directly concern the general theory of Banach spaces; however, they are closely related to theorem 5.2.

5.9. *There exists a continuous real function f defined on the square* $[0,1]\times[0,1]$ *which cannot be uniformly approximated by functions of the form*

$$g(s,t) = \sum_{j=1}^{n} a_j f(s, t_j) f(s_j, t)$$

where $a_1, \ldots, a_n$, are arbitrary real numbers, $s_1, \ldots, s_n, t_1, \ldots, t_n$, belong to the interval $[0,1]$, *and* $n = 1, 2, \ldots$

5.10. *We have*

(a) *For every real β with $2/3 < \beta \leqslant 1$ there exists a real matrix* $A = (a_{ij})_{i,j=1}^{\infty}$ *such that*

$$(+) \qquad A^2 = 0, \quad \text{i.e.} \quad \sum_{j=1}^{\infty} a_{ij} a_{jk} = 0 \quad \text{for} \quad i, k = 1, 2, \ldots,$$

$$(++) \qquad \sum_{j=1}^{\infty} \sup_i |a_{ij}|^{\beta} < \infty,$$

$$(+++) \qquad \sum_{i=1}^{\infty} a_{ii} \neq 0.$$

(b) *If a matrix $A = (a_{ij})$ satisfies $(+)$ and $(++)$ with $\beta = 2/3$, then* $\sum_{i=1}^{\infty} a_{ii} = 0$.

Grothendieck [4] has proved that 5.9 and 5.10 (a) for $\beta = 1$ are equivalent to the existence of a Banach space not having the ap. (The implication "5.9 $\Rightarrow$ 5.2 for $p = \infty$" was already known to Mazur around the year 1936.) 5.10 (a) for $2/3 < \beta < 1$ was observed by Davie [3]. 5.10 (b) is due to Grothendieck [4].

Finally note that there are uniform algebras (Milne [1]) and Banach lattices (Szankowski [3]) which fail to have ap.

§ 6. The bounded approximation property

In general, a proof that a particular Banach space has the approximation property shows that the space in question already has a stronger property. Several properties of that type are discussed by Lindenstrauss [1], Johnson, Rosenthal and Zippin [1], Grothendieck [4] and Pełczyński and Rosenthal [1]. Here we shall only discuss the bounded approximation property, and in the next section the existence of a Schauder basis.

Definition. A Banach space Y is said to *have the* bap (= *the bounded approximation property*) if there exists a constant $a \geqslant 1$ such that, for every $\varepsilon > 0$ and for every compact set $C \subset Y$, there exists an $F \in F(X, X)$ such that

$$(*) \qquad \|Fx - x\| < \varepsilon \text{ for } x \in C \quad \text{and} \quad |||F||| \leqslant a.$$

More precisely, we then say that Y has the bap with a constant a.

It is not difficult to show that

6.1. *A separable Banach space Y has the* bap *if and only if there exists a sequence (F_n) of finite rank operators such that*

$$\lim_n \|F_n y - y\| = 0 \quad \textit{for all } y \in Y.$$

From 5.1 we immediately get

6.2. *If a Banach space has the* bap, *then it has the* ap.

Figiel and Johnson [1] have shown that the converse of 6.2 is not true.

6.3. *There exists a Banach space FJ which has the* ap *but fails the* bap.

The idea of the proof of 6.3 is the following. Let X be a Banach space with the bap and such that X^* does not have the ap, for instance let $X = BJ$ of 5.8. Next we make use of the following lemma:

6.4. *Let Y be a Banach space and let $a \geqslant 1$. If every Banach space isomorphic to Y has the* bap *with the constant a, then Y^* has the* bap.

It follows from 6.4 that, for every positive integer n, there exists a Banach space X_n isomorphic to X and such that X_n does not have the bap with any constant a less than n. We put

$$FJ = (X_1 \times X_2 \times \ldots)_{l^2}.$$

Clearly, every isomorphic image of a space having the ap has the ap. Thus each X_n has the ap. Hence, by 5.5, the space FJ has the ap. On the other hand, FJ fails the bap. This follows from the fact that if a Banach space Y has the bap with a constant a and if Z is a subspace of Y which is the range of a projection of norm $\leqslant 1$, then Z has the bap with a constant $\leqslant a$.

The space FJ also has the following interesting property:

6.5. *There is no sequence (K_n) of compact linear operators such that $\lim_n \|K_n x - x\| = 0$ for all $x \in FJ$.*

Indeed, the existence of such a sequence combined with the fact that FJ has the ap would imply the existence of a sequence (F_n) of finite rank operators such that $|||F_n - K_n||| \leqslant 2^{-n}$ for $n = 1, 2, \ldots$ Hence we would have $\lim_n \|F_n x - x\| = 0$ for all $x \in X$, which, by 6.1, would contradict the fact that the space FJ does not have the bap.

The result 6.5 answers in the negative a question raised in [B], Rem. VI, § 1, p. 209.

Freda Alexander [1] has observed that, for $p > 2$, there exists a subspace X_p of the space L_p such that $F(X_p, X_p)$ is not dense (in the norm topology) in $K(X_p, X_p)$.

Example 6.3 of Figiel and Johnson contrasts with the following deep result (Grothendieck [4], cf. Lindenstrauss–Tzafriri [1] for a simple proof).

6.6. *If a Banach space X is either reflexive or separable and conjugate to a Banach space and if X has the* ap, *then X has the* bap.

Next observe that the improved Local Reflexivity Principle 3.11 yields an analogue of 5.6.

6.7 (Grothendieck [4]). *If X is a Banach space such that X^* has the* bap *with a constant a, then X has the* bap *with a constant $\leqslant a$.*

We conclude this section with a result which gives a characterization of the bounded approximation property in an entirely different language.

Let S be a closed subset of a compact metric space T and let E and X be closed linear subspaces of the spaces $C(S)$ and $C(T)$, respectively. The pair (E, X) is said to *have the bounded extension property*, if, given $\varepsilon > 0$, every function $f \in E$ has a bounded family of extensions

$$\Phi(f, \varepsilon) = \{f_{\varepsilon, W}\colon W \supset S,\ W \text{ is open in } T\} \subset X$$

such that $|f_{\varepsilon, W}(t)| \leqslant \varepsilon$ whenever $t \in T \setminus W$.

6.8. *For every separable Banach space Y the following conditions are equivalent:*

(i) *Y has the* bap,

(ii) *for every closed subset of a compact metric space T, for every isometrically isomorphic embedding $i\colon Y \to C(S)$ and for every closed linear subspace X of the space $C(T)$ such that the pair $\big(i(Y), X\big)$ has the bounded extension property, there exists a bounded linear operator $L\colon i(Y) \to X$ such that $(Lf)(s) = f(s)$ for $s \in S$ and $f \in i(Y)$.*

The proof of the implication (i)$\Rightarrow$(ii) is due to Ryll-Nardzewski, cf. Pełczyński and Wojtaszczyk [1] and Michael and Pełczyński [1]. The implication (ii)$\Rightarrow$(i) has been established by Davie [2].

§ 7. Bases and their relation to the approximation property

The bounded approximation property is closely connected with the property of the existence of a basis in the space. Recall that a sequence (e_n) of elements of a Banach space X constitutes a *basis* for X if, for every $x \in X$, there exists a unique sequence of scalars $(f_n(x))$ such that

$$x = \sum_{n=1}^{\infty} f_n(x) e_n.$$

The map $x \to f_n(x)$ is a continuous linear functional on X called the *n-th coefficient functional* of the basis (e_n) ([B], Chap. VII, § 3). Let us set

$$S_n(x) = \sum_{m=1}^{n} f_m(x) e_m \quad \text{for} \quad x \in X;\ n = 1, 2, \ldots$$

Clearly (S_n) is a sequence of finite rank projections with the property: $\lim_n \|S_n(x) - x\| = 0$ for $x \in X$. Thus, by 6.1, we get

7.1. *If a Banach space X has a basis, then X is separable and has the bounded approximation property.*

Hence every example of a separable Banach space which fails the bap provides an example of a separable Banach space which does not have any basis. No example of a Banach space which has the bap and does not have any basis is known.

On the other hand, we have also a "positive" result relating the bap and the existence of a basis.

7.2. *A separable Banach space has the* bap *if and only if it is isomorphic to a complemented subspace of a Banach space with a basis.*

This has been established by Johnson, Rosenthal and Zippin [1] and Pełczyński [6].

Let us mention some theorems related to 7.2.

7.3 (Lindenstrauss [5], Johnson [1]). *Let X be a separable conjugate (resp. separable reflexive) Banach space. Then X has the* bap *if and only if X is isomorphic to a complemented subspace of a separable conjugate (resp. reflexive) space with a basis.*

Note that, by 6.6, one can replace in 7.3 the "bap" by the "ap".

7.4. *There exists a Banach space UB, unique up to an isomorphism, with a basis (e_n) with the coefficient functionals (f_n) such that:*

(a) *every separable Banach space with the* bap *is isomorphic to a complemented subspace of UB;*

(b) *for every basis* (y_k) *of a Banach space* Y, *there exist an increasing sequence* (m_k) *of indices, an isomorphic embedding* $T\colon Y \to UB$ *and a projection* $P\colon UB \to T(Y)$ *such that* $Ty_k = \|y_k\| e_{n_k}$ *for* $k = 1, 2, \ldots$ *and* $P(x) = \sum_{k=1}^{\infty} f_{n_k}(x) e_{n_k}$ *for* $x \in UB$.

Part (b) has been proved by Pełczyński [8]. (a) follows from (b) via 7.2. Schechtman [2] gave a simple proof of 7.3 (b). Johnson and Szankowski [1], completing 7.3 (a), have shown that if E is a Banach space such that every separable Banach space with ap is isomorphic to a complemented subspace of E, then E is not separable.

Still open question is "the finite-dimensional basis problem". For a basis (e_n) with the coefficient functionals (f_n), we put

$$K(e_n) = \sup_m \sup_{\|x\| \leqslant 1} \Big\| \sum_{n=1}^{m} f_n(x) e_n \Big\|.$$

Next, if X is a Banach space with a basis, we let $K(X) = \inf K(e_n)$ where the infimum is taken over all bases for X. Finally, we define

$$K^{(n)} = \sup \{K(X)\colon \dim X = n\}.$$

The finite-dimensional basis problem is the following: Is it true that $\lim_n K^{(n)} = \infty$.

It is easy to show that $K^{(2)} = 1$ and it is known that $K^{(n)} > 1$ for $n > 2$ (Bohnenblust [2]). It follows from John's theorem 1.1 that $K^{(n)} \leqslant n^{1/2}$. Enflo [4] has proved that there exists a Banach space X isomorphic to the Hilbert space l^2 and such that $K(X) > 1$. Using 7.2 it is easy to show that Johnson's space BJ of 5.8 has a basis. Thus, by 6.4, we infer that, for each n, there exists a Banach space X_n (isomorphic to BJ) with a basis and such that $K(X_n) \geqslant n$.

In the same way as for the ap and bap we have

7.5 (Johnson, Rosenthal and Zippin [1]). *If* X^* *has a basis, then so does* X. *Conversely, if* X *has a basis,* X^* *is separable and has the* ap, *then* X^* *has a basis.*

On the other hand, it follows from Lindenstrauss [5] that there exists a Banach space Z with a basis such that Z^* is separable and fails the ap, and hence Z^* does not have any basis.

For the most common Banach spaces bases have been constructed. We mention here two results of this nature.

7.6 (Johnson, Rosenthal and Zippin [1]). *If* X *is a separable Banach space such that either* X *or* X^* *is isomorphic to a complemented subspace of a space* E *which is either* C *or* L^p $(1 \leqslant p < \infty)$, *then* X *has a basis.*

Let Ω be a compact finite-dimensional differentiable manifold with or without boundary. Denote by $C^k(\Omega)$ the Banach space of all real functions on Ω which have all continuous partial derivatives of order $\leqslant k$.

7.7. *The space $C^k(\Omega)$ has a basis.*

In particular, for $\Omega = [0,1]\times[0,1]$ and $k = 1$, we obtain a positive answer to the question ([B], Rem. VII, § 3, p. 209) whether the space $C^1([0,1]\times[0,1])$ has a basis.

The proof of 7.7 is reduced to the case of concrete manifolds by the following result of Mityagin [3]:

7.8. *For a fixed pair (k, n) of natural numbers, if Ω_1 and Ω_2 are n-dimensional differentiable manifolds with or without boundary, then the spaces $C^k(\Omega_1)$ and $C^k(\Omega_2)$ are isomorphic.*

Now 7.7 follows from Ciesielski [1], Ciesielski and Domsta [1], and independently from Schonefeld [1], [2], where explicit constructions of bases in $C^k(\Omega)$ are given, for Ω being either the n-cube $[0, 1]^n$ or the n-torus T^n $(n, k = 1, 2, \ldots)$.

Bočkariev [1] answering a question of [B], Rem. VII, § 3, p. 209, has shown that the Disc Algebra = the space of [B], Example 10, p. 32 has a basis.

The theorem of Banach stating that

7.9. *Every infinite-dimensional Banach space contains an infinite-dimensional subspace with a basis;*

and announced in [B], Rem. VII, § 3, p. 209, has been improved and modified in several papers (cf. Bessaga and Pełczyński [3], [4], Day [5], Gelbaum [1], Davis and Johnson [2], Johnson and Rosenthal [1], Kadec and Pełczyński [2], Milman [1], Pełczyński [7]). In particular, it has been shown that

7.10 (Pełczyński [7]). *Every non-reflexive Banach space contains a non-reflexive subspace with a basis.*

7.11 (Johnson and Rosenthal [1]). *Every infinite Banach space which is the conjugate of a separable Banach space contains an infinite-dimensional subspace which has a basis and which is a conjugate space.*

7.12 (Johnson and Rosenthal [1]). *Every separable infinite-dimensional Banach space admits an infinite-dimensional quotient with a basis.*

The separability assumption in 7.12 is related to the open question whether every Banach space has a separable infinite-dimensional quotient.

There is a huge literature concerning the classification of bases and their generalizations, and also concerning the properties of special bases. The reader may consult the books by Day [1], Lindenstrauss and Tzafriri [1], Singer [1] and the surveys by Milman [1] and McArthur [1], where bases in Banach spaces are discussed, the book by Rolewicz [2] and the surveys by Dieudonné [2], [3], Mityagin [1], [2] and McArthur [1], where bases in general linear topological spaces are treated.

Concluding this section, we add that the question raised in [B] Rem. VII, § 1, p. 209 has been answered by Ovsepian and Pełczyński [1]. We have (cf. Pełczyński [9])

7.13. *Every separable Banach space X admits a biorthogonal system* (x_n, f_n) *such that* $\|x_n\| = 1$ *for* $n = 1, 2, \ldots$, $\lim_n \|f_n\| = 1$, *and* (a) *if* $f \in X^*$ *and* $f(x_n) = 0$ *for all n, then* $f = 0$, *and* (b) *if* $x \in X$ *and* $f_n(x) = 0$ *for all n, then* $x = 0$. *Moreover, given* $c > 1$ *the biorthogonal sequence can be chosen so that* $\sup_n \|f_n\| < c$.

It is unknown whether the "Moreover" part of 7.13 is true for $c = 1$.

§ 8. Unconditional bases

A basis (e_n) for a Banach space X is *unconditional* if

$$\sum_{n=1}^{\infty} |f_n(x) x^*(e_n)| < \infty \qquad \text{for all } x \in X;\ x^* \in X^*,$$

where (f_n) is the sequence of coefficient functionals of the basis (e_n).

The existence of an unconditional basis in the space is a very strong property. It determines on the space the Boolean algebra of projections (P_σ), where, for any subset σ of positive integers, the projection $P_\sigma \in B(X, X)$ is defined by

$$P_\sigma(x) = \sum_{n \in \sigma} f_n(x) e_n,$$

and, in the real case, it determines also the lattice structure on X induced by the partial ordering: $x \prec y$ iff $f_n(x) \leqslant f_n(y)$ for $n = 1, 2, \ldots$

Several results on unconditional bases can be generalized to an arbitrary Boolean algebra of projections, and Banach lattices. The reader is referred to Dunford and Schwartz [1], Part III, Lindenstrauss and Tzafriri [1].

To illustrate the consequences of the existence of an unconditional basis in a Banach space, we state an already classical result due to R. C. James [1].

8.1. *A Banach space with an unconditional basis is reflexive if and only if none of its subspaces is isomorphic either to* c_0 *or to* l^1.

From 8.1, 1.5 and 1.6 it immediately follows that the spaces J and DJ defined in § 1 have no unconditional bases. In fact, these spaces cannot be isomorphically embedded into any Banach space with an unconditional basis. Therefore the universal space C ([B], Chap. XI, § 8) has no unconditional basis.

The existence of unconditional bases in sequence spaces like l^p $(1 \leqslant p < \infty)$, c_0 and in separable Orlicz sequence spaces (= the space (o) in the notation of [B], Rem. Introduction, § 7, p. 201) is trivial. The next result of Paley [2] and Marcinkiewicz [1] is much more difficult.

8.2. *The Haar system is an unconditional basis in the spaces* L^p *for* $1 < p < \infty$.

For a relatively simple proof of this theorem see Burkholder [1].

The Paley–Marcinkiewicz theorem can be generalized to symmetric function spaces. A *symmetric function space* is a Banach space E consisting of equivalence classes of Lebesgue measurable functions on $[0, 1]$ such that

(a) $L^\infty \subset E \subset L^1$,

(b) if $f_1 \in E$, f_2 is a measurable function on $[0, 1]$ such that if $|f_2|$ is equidistributed with $|f_1|$, then $f_2 \in E$, and $\|f_2\|_E = \|f_1\|_E$.

The following result is due to Olevskiĭ [1], cf. Lindenstrauss and Pełczyński [2] for a proof.

8.3. *A symmetric function space E has an unconditional basis if and only if the Haar system is an unconditional basis for E.*

Combining 8.2 with the interpolation theorem of Semenov [1], we get

8.4. *Let E be a symmetric function space and let* $g_E(t) = \|\chi_{[0,t]}\|_E$, *where* $\chi_{[0,t]}$ *denotes the characteristic function of the interval* $[0, t]$. *If* $1 < \liminf_{t\to 0} g_E(2t)/g_E(t) \leqslant \limsup_{t\to 0} g_E(2t)/g_E(t) < 2$, *then the Haar system is an unconditional basis for E.*

A corollary to this theorem is the following result, established earlier in a different way by Gaposhkin [1]:

8.5. *An Orlicz function space (= the space (O) in the notation of* [B], *pp.* 202–203) *has an unconditional basis if and only if it is reflexive.*

An important class of unconditional bases is that of symmetric bases. A basis (e_n) for X with the sequence of coefficient functionals (f_n) is called *symmetric* if, for every $x \in X$ and for every permutation $p(\cdot)$ of the indices, the series $\sum_{n=1}^{\infty} f_n(x) e_{p(n)}$ converges.

The next result is due to Lindenstrauss [9].

8.6. *Let* (y_k) *be an unconditional basis in a Banach space Y. Then there exist a symmetric basis* (x_n) *in a Banach space X and an isomorphic embedding* $T\colon Y \to X$ *whose values on the vectors* y_k *are*

$$Ty_k = c_k \cdot \sum_{n_k < n \leqslant n_{k+1}} x_n \quad \textit{for} \quad k = 1, 2, \ldots,$$

for some scalars c_k *and indices* $1 \leqslant n_1 < n_2 < \ldots$

For every symmetric basis (e_n) with the coefficient functionals f_n $(n = 1, 2, \ldots)$ and for every increasing sequence of indices (n_k), the operator $P\colon X \to X$ defined by

$$P(x) = \sum_{k=1}^{\infty} \big((n_{k+1} - n_k)!\big)^{-1} \cdot \sum_{p \in \Pi_k} \sum_{j=n_k+1}^{n_{k+1}} f_{p(j)}(x) e_j,$$

where Π_k denotes the set of all permutations of the indices $n_k+1, \ldots, n_{k+1}$, is a bounded projection onto the subspace of X spanned by the blocks $\sum_{j=n_k+1}^{n_{k+1}} e_j$ $(k = 1, 2, \ldots)$. Hence, by 8.6, we have

8.7 (Lindenstrauss [9]). *Every Banach space with an unconditional basis is isomorphic to a complemented subspace of a Banach space with a symmetric basis.*

It is not known whether the converse of 8.7 is true or, equivalently, whether every complemented subspace of a Banach space with an unconditional basis has an unconditional basis. The question is open even for complemented subspaces of L^p $(1 < p < \infty;\ p \neq 2)$.

The next result is similar to 7.4.

8.8. *There exists, a unique up to an isomorphism, Banach space US, with a symmetric basis such that every Banach space with an unconditional basis is isomorphic to a complemented subspace of US. Moreover, the space US has an unconditional but not symmetric basis* (e_n) *with the following property*:

(*) *for every unconditional basis* (y_k) *in any Banach space Y, there exist an isomorphic embedding* $T\colon Y \to US$ *and an increasing sequence of indices* (n_k) *such that* $Ty_k = \|y_k\|\, e_{n_k}$ *for* $k = 1, 2, \ldots$

The existence of an unconditional basis with property (*) has been established by Pełczyński [8], see also Zippin [2] for an alternative simpler proof. Combining (*) with 8.7 one gets the first statement of 8.8.

In contrast to 7.5, we have

8.9. *There exists a Banach space X which does not have any unconditional basis, but its conjugate* X^* *does.*

An example of such a space is $C(\omega^\omega)$, the space of all scalar-valued continuous functions on the compact Hausdorff space of all ordinals $\leqslant \omega^\omega$, whose conjugate is l^1 (cf. Bessaga and Pełczyński [2], p. 62 and Lindenstrauss and Pełczyński [1], p. 297). The existence of a Banach lattice without ap (Szankowski [3]) yields that $(US)^*$ fails to have ap. (However, if X^* is separable and X has an unconditional basis, then X^* also has an unconditional basis!)

We do not know whether every infinite-dimensional Banach space contains an infinite-dimensional subspace with an unconditional basis (compare with 7.9).

We shall end this section with the discussion of the "unconditional finite-dimensional basis problem", which has been solved by Y. Gordon and D. Lewis. For an unconditional basis (e_n) with the coefficient functionals (f_n), we let

$$K_u(e_n) = \sup\Big\{\sum_n |f_n(x)x^*(e_n)|\colon\ \|x\| \leqslant 1,\ \|x^*\| \leqslant 1\Big\}.$$

Next, if X is a Banach space with an unconditional basis, we set $K_u(X) = \inf K_u(e_n)$, where the infimum is taken over all unconditional bases for X. Finally, we define

$$K_u^{(n)} = \sup\{K_u(X)\colon \dim X = n\}.$$

Let $B_n = B(l_n^2, l_n^2)$, the n^2 dimensional Banach space of all linear operators from the n-dimensional Euclidean space into itself.

Gordon and Lewis [1] have proved that

8.10. *There exists a* $C > 0$ *such that* $C\sqrt{n} \leqslant K_u(B_n) \leqslant \sqrt{n}$, *for* $n = 1, 2, \ldots$

In fact, they have obtained a slightly stronger result:

8.11. *If* Y *is a Banach space with an unconditional basis and* Y *contains a subspace isometrically isomorphic to* B_n, *then, for every projection* P *of* Y *onto this subspace, we have*

$$\|P\| \cdot K_u(Y) \geqslant C\sqrt{n},$$

where $C > 0$ *is a universal constant independent of* n.

The exact rate of growth of the sequence $(K_u^{(n)})$ has recently been found by Figiel, Kwapień and Pełczyński [1] who proved that $K_u^{(n)} \geqslant C\sqrt{n}$. It follows from John's Theorem 2.2 that $K_u^{(n)} \leqslant \sqrt{n}$.

CHAPTER IV

§ 9. Characterizations of Hilbert spaces in the class of Banach spaces

The problems concerning isometric and isomorphic characterizations of Hilbert spaces in the class of Banach spaces, posed in [B], pp. 213–214, have stimulated the research activity of numerous mathematicians. Isomorphic characterizations of Hilbert spaces have proved to be much more difficult than the isometric characterizations.

We say that a property (P) *isometrically* (*isomorphically*) *characterizes Hilbert spaces in the class of Banach spaces* if the following statement is true: "A Banach space X has property (P) iff X is isometrically isomorphic (is isomorphic) to a Hilbert space". By a *Hilbert space* we mean any Banach space H (separable, non-separable, or finite-dimensional) whose norm is given by $\|x\| = (x, x)^{1/2}$, where $(\cdot, \cdot)$: $H \times H \to K$ is an inner product and K is the field of scalars (real or complex numbers).

We shall first discuss isometric characterizations of Hilbert spaces. Results in this field are extensively presented in Day's book [1], Chap. VII, § 3. Therefore here we shall restrict ourselves to discussing the most important facts and giving supplementary information.

The basic isometric characterization of Hilbert spaces is due to Jordan and von Neumann [1].

9.1. *A Banach space X is isometrically isomorphic to a Hilbert space iff it satisfies the parallelogram identity*:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \textit{for all } x, y \in X.$$

As an immediate corollary of 9.1 we get

9.2. *A Banach space X is isometrically isomorphic to a Hilbert space if and only if every two-dimensional subspace of X is isometric to a Hilbert space.*

An analogous characterization but with 2-dimensional subspaces replaced by 3-dimensional ones was earlier discovered by Fréchet [1]. In the thirties Aronszajn [1] found other isometric characterizations of a Hilbert space, which, as 9.2, are of a two-dimensional character, i.e. are stated in terms of properties of a pair of vectors in the space.

A characterization of an essentially 3-dimensional character was given by Kakutani [1] (see also Phillips [1]) in the case of real spaces, and by Bohnenblust [1] in the complex case. It states that

9.3. *For a Banach space X with* $\dim X \geqslant 3$ *the following statements are equivalent:*

(i) *X is isometrically isomorphic to a Hilbert space,*

(ii) *every 2-dimensional subspace of X is the range of a projection of norm* 1,

(iii) *every subspace of X is the range of a projection of norm* 1.

Here and in the sequel, by "dim" we mean the algebraic dimension with respect to the corresponding field of scalars.

Assume that H is a Hilbert space with $2 < \dim H \leqslant \infty$ and $2 \leqslant k < \dim H$. Obviously all k-dimensional subspaces of H are isometrically isomorphic to each other. The question ([B], Rem. XII, p. 214, properties (4) and (5)) whether the property above characterizes Hilbert spaces has been solved only partially, i.e. under certain dimensional restrictions. Let us say that a real (resp. complex) Banach space X *has the property* H^k, for $k = 2, 3, \ldots$, if $\dim X \geqslant k$ and all subspaces of X of real (resp. complex) dimension k are isometrically isomorphic to each other.

9.4. *The following two tables give the dimensional restrictions on Banach spaces X under which the property* H^k *implies that X is isometrically isomorphic to a Hilbert space:*

The real case

k even	$k+1 \leqslant \dim X \leqslant \infty$
k odd	$k+2 \leqslant \dim X \leqslant \infty$

The complex case

k even	$k+1 \leqslant \dim X \leqslant \infty$
k odd	$2k \leqslant \dim X \leqslant \infty$

The real case of $k = 2, \dim X < \infty$, was solved by Auerbach, Mazur and Ulam [1]. The case of $\dim X = \infty$ is a straightforward consequence of Dvoretzky's [2] theorem on almost spherical sections (see 3.5). This was observed in Dvoretzky [1]. The remaining statements are due to Gromov [1]. The simplest unsolved case is $k = 3, \dim X = 4$.

We shall mention two more isometric characterizations of Hilbert space.

9.5 (Foiaş [1], von Neumann [1]). *A complex Banach space X is isometrically isomorphic to a Hilbert space if and only if, for every linear operator* $T\colon X \to X$ *and for every polynomial P with complex coefficients, the inequality* $\|P(T)\| \leqslant \|T\| \cdot \sup_{|z|=1} |P(z)|$ *holds.*

9.6. (Auerbach [1], von Neumann [2]). *A finite-dimensional Banach space X is isometrically isomorphic to a Hilbert space if and only if the group of linear isometries of X acts transitively on the unit sphere of X, i.e. for every* *pair of points* $x, y \in X$ *such that* $\|x\| = \|y\| = 1$, *there is a linear isometry* T: $X \xrightarrow[\text{onto}]{} X$ *such that* $T(x) = y$.

Remark. Let $1 \leqslant p < \infty$ and let μ be an arbitrary non-sigma-finite non-atomic measure. Then the group of linear isometries of the space $L^p(\mu)$ acts transitively on the unit sphere of the space. Therefore the assumption of 9.6 that X is finite-dimensional is essential. The question whether there exists a separable Banach space other than a Hilbert space whose group of linear isometries acts transitively on the unit sphere remains open (cf. [B], Rem. XI, § 5, p. 212).

Now we shall discuss various isomorphic characterizations of a Hilbert space. The simplest among them reflects the fact that all subspaces of a fixed dimension of a Hilbert space are isometric, and hence are "equi-isomorphic". More precisely, we have

9.7. *For every Banach space* X *the following statements are equivalent*:

(1) *X is isomorphic to a Hilbert space,*

(2) $$\sup_n \sup_{E \in \mathfrak{A}_n(X)} d(E, l_n^2) < \infty,$$

(3) $$\sup_n \sup_{E \in \mathfrak{A}^n(X)} d(E, l_n^2) < \infty,$$

where $\mathfrak{A}_n(X)$ (*resp.* $\mathfrak{A}^n(X)$) *denotes the family of all n-dimensional subspaces* (*resp. quotient spaces*) *of the space* X.

From the theorem of Dvoretzky, it follows that conditions (2) and (3) can be replaced, respectively, by

(2′) $$\sup_n \sup_{E,F \in \mathfrak{A}_n(X)} d(E, F) < \infty,$$

(3′) $$\sup_n \sup_{E,F \in \mathfrak{A}^n(X)} d(E, F) < \infty,$$

Theorem 9.7 is implicitly contained in Grothendieck [5]. The equivalence between (1) and (2) was explicitly stated by Joichi [1], cf. here 3.1. In connection with 9.7 note that the following question is still unanswered: "If X is a Banach space and all infinite-dimensional subspaces of X are isomorphic to each other, is X then isomorphic to a Hilbert space?" ([B], Rem. XII, p. 214).

The following elegant result of Lindenstrauss and Tzafriri [3] (cf. also Kadec and Mityagin [1]) is an isomorphic analogue of theorem 9.3.

9.8. *A Banach space* X *is isomorphic to a Hilbert space if and only if*:

(*) *each subspace of* X *is complemented.*

This theorem shows that property (7) discussed in [B] on pp. 213–214 is a feature of Banach spaces isomorphic to a Hilbert space only.

The proof of 9.8 starts with an observation of Davis, Dean and Singer [1] that condition (*) implies

$$\infty > \sup_n P_n(X) = \sup_{E\in\mathfrak{A}_n(X)} \inf\{\|P\|: P \text{ is a projection of } X \text{ onto } E\}.$$

Next, by an ingenious use of Dvoretzky's Theorem 3.4, it is shown that $\sup_n P_n(X) < \infty$ implies condition (2) of 9.7.

Historical remark. Theorem 9.8 states that every Banach space which is not isomorphic to any Hilbert space has a non-complemented subspace. The construction of such subspaces in concrete Banach spaces was relatively difficult. Banach and Mazur [1] showed that every isometrical isomorph of l^1 in the space C is not complemented. Murray [1] constructed non-complemented subspaces in the spaces L^p. For a large class of Banach spaces with a symmetric basis an elegant construction of non-complemented subspaces was given by Sobczyk [2].

Combining 9.8 with earlier results of Grothendieck [4], we obtain

9.9. *The only, up to an isomorphism, locally convex complete linear metric spaces with property* (*) *are the Hilbert spaces, the space* s *of all scalar sequences, and the product* $s\times H$, *where* H *is an infinite-dimensional Hilbert space.*

In the same way as 9.8 one can prove (cf. Lindenstrauss and Tzafriri [3])

9.9. *A Banach space* X *is isomorphic to a Hilbert space if and only if, for every subspace* Y *of* X *and for every compact linear operator* $T: Y\to Y$, *there exists a linear operator* $\tilde{T}: X\to Y$ *which extends* T.

An interesting characterization of a Hilbert space is due to Grothendieck [5] (cf. also Lindenstrauss and Pełczyński [1]).

9.10. *A Banach space* X *is isomorphic to a Hilbert space if and only if*:

(**) *there is a constant* K *such that, for every scalar matrix* $(a_{ij})_{i,j=1}^n$ ($n = 1, 2, \ldots$) *and every* $x_1, \ldots, x_n \in X$ *of norm* 1, $x_1^*, \ldots, x_n^* \in X^*$ *of norm* 1, *there are scalars* $s_1, \ldots, s_n, t_1, \ldots, t_n$ *each of absolute value* $\leqslant 1$ *such that*

$$\Big|\sum_{i,j} a_{ij} x_i^*(x_j)\Big| \leqslant K\Big|\sum_{i,j} a_{ij} s_i t_j\Big|.$$

In contrast to the previous characterizations, it is not easy to show that Hilbert spaces have property (**). Interesting proofs of this fact were recently given by Maurey [1], Maurey and Pisier [1], Krivine [3].

Closely related to 9.10 is the following characterization (cf. Grothendieck [5], Lindenstrauss and Pełczyński [1]).

9.11. *A separable Banach space* X *is isomorphic to a Hilbert space iff* X *and* X^* *are isomorphic to subspaces of the space* L^1 *iff* X *and* X^* *are isomorphic to quotient spaces of* C.

In the above theorem the assumption of separability of X can be dropped if one replaces the spaces L^1 and C by "sufficiently big" $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces. (For the definition see section 10.)

Let us notice that every separable Hilbert space is isometrically isomorphic to a subspace of L^1 (cf. e.g. Lindenstrauss and Pełczyński [1]). We do not know whether 9.11 admits an isometrical version, i.e. whether every infinite dimensional Banach space X such that X and X^* are isometrically isomorphic to subspaces of L^1 is isometrically isomorphic to a Hilbert space. For partial results see Bolker [1]. For $\dim X < \infty$ the answer is negative (R. Schneider [1]).

From the parallelogram identity one obtains by induction, for $n = 2, 3, \ldots$ and for arbitrary elements of a Hilbert space,

$$2^{-n} \sum_{\varepsilon} \|\varepsilon_1 x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_n x_n\|^2 = \sum_{j=1}^{n} \|x_j\|^2,$$

where $\sum_{\varepsilon}$ denotes the sum extended over all sequences $(\varepsilon_1, \ldots, \varepsilon_n)$ of ± 1's. The following isomorphic characterization of Hilbert spaces, due to Kwapień [1], is related to the above identity.

9.12. *A Banach space X is isomorphic to a Hilbert space if and only if there exists a constant A such that*

$$A^{-1} \sum_{j=1}^{n} \|x_j\|^2 \leqslant \sum_{\varepsilon} \Big\| \sum_{j=1}^{n} \varepsilon_j x_j \Big\|^2 \leqslant A \sum_{j=1}^{n} \|x_j\|^2$$

for arbitrary $x_1, \ldots, x_n \in X$ and for $n = 2, 3, \ldots$

From 9.12 Kwapień [1] has derived another isomorphic characterization of Hilbert spaces. In order to state it, we shall need some additional notation. Let $L_0^2(R, X)$ denote the normed linear space consisting of simple functions with values in the Banach space X and with supports of finite Lebesgue measure in R. We define $|f| = \big(\int_{-\infty}^{+\infty} \|f(t)\|^2 dt\big)^{1/2}$ for $f \in L_0^2(R, X)$. By $L^2(R, X)$ we denote the completion of $L_0^2(R, X)$ in the norm $|\cdot|$. The Fourier transformation F: $L_0^2(R, X) \to L^2(R, X)$ is defined by the classical formula

$$F(f)(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-ist} f(s)\, ds.$$

Under this notation we have

9.13. *For every complex Banach space X the following statements are equivalent:*

(i) *X is isomorphic to a Hilbert space.*

(ii) *There is a constant $A > 0$ such that*

$$\sum_{j=-n}^{n} \|x_j\|^2 \leqslant A \int_0^{2\pi} \Big\| \sum_{j=-n}^{n} e^{jit} x_j \Big\|^2 dt$$

for arbitrary $x_{-n}, \ldots, x_0, \ldots, x_n \in X$ and for $n = 1, 2, \ldots$

(iii) *There exists a constant $A > 0$ such that*

$$\int_0^{2\pi} \Big\| \sum_{j=-n}^{n} e^{ijt} x_j \Big\|^2 dt \leqslant A \sum_{j=-n}^{n} \|x_j\|^2$$

for arbitrary $x_{-n}, \ldots, x_0, \ldots, x_n \in X$ and for $n = 1, 2, \ldots$

(iv) *The Fourier transformation $F\colon L_0^2(R, X) \to L^2(R, X)$ is a bounded linear operator.*

Using 9.12 Figiel and Pisier [1] have proved that

9.14. *A Banach space X is isomorphic to a Hilbert space if and only if there exist a constant $A > 0$ and Banach spaces X_1 and X_2 isomorphic to X such that X_1 is uniformly convex, X_2 is uniformly smooth and the moduli of convexity and smoothness satisfy the inequalities $\delta_{X_1}(t) \geqslant At^2$, $\varrho_{X_2}(t) \leqslant At^2$ for small $t > 0$.*

Meškov [1] improving a result of Sundaresan [1] has shown that

9.15. *A real Banach space X is isomorphic to a Hilbert space if and only if X and X^* equivalent norms which are twice differentiable everywhere except the origins of X and X^*.*

An operator $T\colon X \to Y$ is *nuclear* if there are $x_j^* \in X^*$, $y_j \in Y$ $(j = 1, 2, \ldots)$ with $\sum_{j=1}^{\infty} \|x_j^*\| \, \|y_j\| < \infty$ and $Tx = \sum_{j=1}^{\infty} x_j^*(x) y_j$ for $x \in X$. P. Ørno observed (cf. Johnson, König, Maurey and Retherford [1])

9.16. *A Banach space X is isomorphic to a Hilbert space iff every nuclear $T\colon X \to X$ has summable eigenvalues.*

Enflo [1] gave a non-linear characterization of Hilbert spaces.

9.17. *A Banach space X is isomorphic to a Hilbert space if and only if X is uniformly homeomorphic to a Hilbert space H, i.e. there is a homeomorphism $h\colon X \xrightarrow[\text{onto}]{} H$ such that h and h^{-1} are uniformly continuous functions in the metrics induced by the norms of X and H.*

CHAPTER V

Classical Banach spaces

The spaces $L^p(\mu)$ and $C(K)$ are distinguished among Banach spaces by their regular properties. However, most of those properties, of both isomorphic and isometric character, extend to some wider classes of spaces, which can easily be defined in terms of finite-dimensional structure, i.e. by requiring certain properties of finite-dimensional subspaces of a given space.

Definition (Lindenstrauss and Pełczyński [1]). Let $1 \leqslant p \leqslant \infty$ and let $\lambda > 1$. A Banach space X is an *$\mathscr{L}_{p,\lambda}$ space* if, for every finite-dimensional subspace $E \subset X$, there is a finite-dimensional subspace $F \subset X$ such that $F \supset E$ and $d(F, l_k^p) < \lambda$, where $k = \dim F$. The space X is an *$\mathscr{L}_p$ space* provided that it is an $\mathscr{L}_{p,\lambda}$ space for some $\lambda \in (1, \infty)$.

The class $\mathscr{L}_p = \bigcup_{\lambda > 1} \mathscr{L}_{p,\lambda}$ is the required class of spaces which have most of the isomorphic properties of the spaces $L^p(\mu)$ and $C(K)$ (for $p = \infty$). From the point of view of the isometric theory the natural class is the subclass of $\mathscr{L}_p$ consisting of all those spaces X which are $\mathscr{L}_{p,\lambda}$ for every $\lambda > 1$, i.e. the class $\bigcap_{\lambda > 1} \mathscr{L}_{p,\lambda}$.

§ 10. The isometric theory of classical Banach spaces

First, we shall discuss the case $1 \leqslant p < \infty$, which is simpler than that of $p = \infty$. We have

10.1. *Let $1 \leqslant p < \infty$. A Banach space X is isometrically isomorphic to an $L^p(\mu)$ space if and only if X is an $\mathscr{L}_{p,\lambda}$ space for every $\lambda > 1$.*

Recall that a projection $P\colon X \to X$ is said to be *contractive* if $\|P\| \leqslant 1$.

10.2. *If P is a contractive projection in a space $L^p(\mu)$, then $Y = P(L^p(\mu))$ is an $\mathscr{L}_{p,\lambda}$ space for every $\lambda > 1$.*

The proofs of 10.1 and 10.2 are due to the combined effort of many mathematicians (for the history see Lacey [1]). They are based in an essential way on the following theorem on the representation of Banach lattices, which (in a less general form) has been discovered by Kakutani and Bohnenblust.

Recall that if x is a vector in a Banach lattice, then $|x|$ is defined to be $\max(x, 0)+\max(-x, 0)$.

10.3. *Let* $1 \leqslant p \leqslant \infty$. *A Banach lattice* X *is lattice-isometrically isomorphic to a Banach lattice* $L^p(\mu)$ *if and only if* $(\|x\|^p+\|y\|^p)^{1/p} = \|x+y\|$ *whenever* $\min(|x|, |y|) = 0$, *for* $x, y \in X$. (*If* $p = \infty$, *then by* $(\|x\|^p+\|y\|^p)^{1/p}$ *we mean* $\max(\|x\|, \|y\|)$).

We also have (Ando [1])

10.4. *If* X *is a Banach lattice with* $\dim X \geqslant 3$, *then* X *is lattice-isometrically isomorphic to a lattice* $L^p(\mu)$ *if and only if every proper sublattice of* X *is the image of a positive contractive projection.*

In particular, if $1 \leqslant p < \infty$, *then every separable subspace of* $L^p(\mu)$ *is contained in a subspace of the space which is isomorphic to a space* $L^p(\nu)$ *and which is the image of a contractive projection.*

For $1 < p < \infty$ the spaces $L^p(\mu)$ are reflexive (and even uniformly convex and uniformly smooth). We have

10.5. $(L^p(\mu))^* = L^{p^*}(\mu)$, *with* $p^* = p/(p-1)$. *The equality means here the canonical isomorphism given by* $f \to \int \cdot f d\mu$ *for* $f \in L^{p^*}(\mu)$.

This is a generalization of the classical theorem of Riesz [1] (cf. [B], p. 72).

Theorem 10.5 remains valid for $p = 1$ ($p^* = \infty$) in the case of sigma-finite measures. For arbitrary measures we have only the following fact (see e.g. Pełczyński [2]):

10.6. *For every measure* μ *there exists a measure* ν (*which in general is defined on another sigma-field of sets*) *such that the spaces* $L^1(\mu)$ *and* $L^1(\nu)$ *are isomorphic and such that the map* $f \to \int \cdot f d\nu$ *is an isometrical isomorphism of* $L^\infty(\nu)$ *onto* $(L^1(\nu))^*$.

The following theorem is due to Grothendieck [2]:

10.7. *If* X^* *is isometrically isomorphic to a space* $C(K)$, *then* X *is isometrically isomorphic to a space* $L^1(\nu)$.

The isometric classification of spaces $L^p(\nu)$ reduces to the Boolean classification of measure algebras (S, Σ, μ). The latter is relatively simple in the case of sigma-finite measures. We have

10.8. *If* μ *is a sigma-finite measure, then the space* $L^p(\mu)$ *is isometrically isomorphic to a finite or infinite product*

$$(l^p(A) \times L^p(\lambda^{n_1}) \times L^p(\lambda^{n_2}) \times \ldots)_p$$

where A *is the set of atoms of the measure* μ *and* $n_1, n_2, \ldots$ *is a sequence of distinct cardinals and* λ^n *denotes the measure which is the product of* n *copies of the measure* λ *defined on the field of all subsets of the two-point set* $\{0, 1\}$ *such that* $\lambda(\{0\}) = \lambda(\{1\}) = 1/2$.

Theorem 10.8 is a consequence of a profound result of Maharam [1] stating that every homogeneous measure algebra is isomorphic to a measure algebra of the measure λ^n for some cardinal n.

From 10.8 and the remark after 10.4 it easily follows that every separable space $L^p(\mu)$ is isometrically isomorphic to the image of a contractive projection in the space L^p (for $1 \leqslant p < \infty$).

Now we shall discuss the case $p = \infty$.

Definition. A Banach space X is called a *Lindenstrauss space* if its dual X^* is isometrically isomorphic to a space $L^1(\mu)$.

The classical theorem of Riesz on the representation of linear functionals on $C(K)$ (for the proof see, for instance, Dunford and Schwartz [1] and Semadeni [2]) combined with theorem 10.3 shows that all the spaces $C(K)$ are Lindenstrauss spaces. It is particularly interesting to note that the class of Lindenstrauss spaces is essentially wider than the class of spaces $C(K)$, for instance c_0 is a Lindenstrauss space which is not isometrically isomorphic to any space $C(K)$. Also, if S is a Choquet simplex (for the definition see Alfsen [1]), then the space $\mathrm{Af}(S)$ of all affine scalar functions on S is a Lindenstrauss space; so is the space in 11.15. Now we state several results.

10.9. *For every Banach space X the following statements are equivalent*:

(1) *X is an $\mathscr{L}_{\infty,\lambda}$ space for every $\lambda > 1$*,

(2) *X is a Lindenstrauss space*,

(3) *the second dual X^{**} is isometrically isomorphic to a space $C(K)$*.

10.10. *A Lindenstrauss space X is isometrically isomorphic to a space $C(K)$ if and only if the unit ball of X has at least one extreme point and the set of extreme points of X^* is w^*-closed.*

Every space $L^\infty(\mu)$ is isometrically isomorphic to a space $C(K)$.

The following is an analogue of 10.2:

10.11. *If P is a contractive projection in a Lindenstrauss space X, then $P(X)$ is a Lindenstrauss space.*

It should be noted that not all Lindenstrauss spaces are images of spaces $C(K)$ under contractive projections (cf. Lazar and Lindenstrauss [1] for details). However, we have

10.12 (Lazar and Lindenstrauss [1]). *Every separable Lindenstrauss space is isometrically isomorphic to the image of a contractive projection in a space* $\mathrm{Af}(S)$.

Grothendieck [4] has observed that in the class of Banach spaces Lindenstrauss spaces can be characterized by some properties of the extension of linear operators, and spaces $L^1(\mu)$ can be characterized by properties of lifting linear operators. We have

10.13. *For every Banach space X the following statements are equivalent*:

(a1) *X is a Lindenstrauss space.*

(a2) *For arbitrary Banach spaces E, F, an isometrically isomorphic embedding $j\colon F \to E$, a compact linear operator $T\colon F \to X$ and $\varepsilon > 0$, there exists a compact linear operator $\tilde{T}\colon E \to X$ which extends T (i.e. $T = \tilde{T}j$) and is such that $\|\tilde{T}\| \leqslant (1+\varepsilon)\|T\|$.*

(a3) *For arbitrary Banach spaces Y, Z, an isometrically isomorphic embedding $j\colon X \to Y$ and a compact linear operator $T\colon X \to Z$ there exists a compact linear operator $\tilde{T}\colon Y \to Z$ such that $T = \tilde{T}j$ and $\|\tilde{T}\| = \|T\|$.*

10.14. *For every Banach space X the following statements are equivalent:*

(a*1) *X is isometrically isomorphic to a space $L^1(\mu)$.*

(a*2) *For an arbitrary Banach space E, its quotient space F, a compact linear operator $T\colon X \to F$ and $\varepsilon > 0$ there exists a compact linear operator $\tilde{T}\colon X \to E$ with $\|\tilde{T}\| \leqslant (1+\varepsilon)\|T\|$ which lifts T, i.e. $T = \varphi\tilde{T}$, where φ is the quotient map of E onto F.*

(a*3) *For arbitrary Banach spaces Y, Z, a linear operator $\varphi\colon Y \xrightarrow[\text{onto}]{} X$ and a compact linear operator $T\colon Z \to X$ there exists a compact linear operator $\tilde{T}\colon Z \to Y$ such that $\|\tilde{T}\| = \|T\|$ and $T = \varphi\tilde{T}$.*

Other interesting characterizations can be found in Lindenstrauss [1], [2].

Omitting in (a2), (a3) (resp. in (a*2), (a*3)) the requirement that the linear operators T and $\tilde{T}$ should be compact, we obtain characterizations of important classes of injective (resp. projective) Banach spaces. They are narrow subclasses of Lindenstrauss spaces (resp. of spaces $L^1(\mu)$); see the theorems below.

Recall that a compact Hausdorff space K is said to be *extremally disconnected* if the closure of every open set in K is open.

10.15 (Nachbin–Goodner–Kelley). *For every Banach space X the following statements are equivalent:*

(b1) *X is isometrically isomorphic to a space $C(K)$ with K extremally disconnected.*

(b2) *For arbitrary Banach spaces E, F, an isometrically isomorphic embedding $j\colon E \to F$, and a linear operator $T\colon E \to X$, there exists a linear operator $\tilde{T}$ such that $T = \tilde{T}j$ and $\|\tilde{T}\| = \|T\|$.*

(b3) *X satisfies* (a2) *with "compact linear operator" replaced by "linear operator".*

(b4) *X satisfies* (a3) *with "compact linear operator" replaced by "linear operator".*

10.16. *For every Banach space X the following statements are equivalent:*

(b*1) *X is isometrically isomorphic to a space $l^1(S)$.*

(b*2) *For an arbitrary Banach space E, its quotient space F and a linear operator $T\colon X \to F$ there exists a linear operator $\tilde{T}\colon X \to E$ such that $\|\tilde{T}\| = \|T\|$ and $T = \varphi\tilde{T}$ where $\varphi\colon E \to F$ is the quotient map.*

(b*3) *X satisfies* (a*2) *with "compact linear operator" replaced by "linear operator".*

(b*4) *X satisfies* (a*3) *with "compact linear operator" replaced by "linear operator".*

The isometrical classification of the spaces $C(K)$ reduces to the topological classification of compact Hausdorff spaces. For compact metric spaces this fact has been established by Banach (see [B], Chap. IX, Théorème 3). The general result is due to M. H. Stone [1] and S. Eilenberg [1]. It is as follows:

10.17. *Compact Hausdorff spaces K_1 and K_2 are homeomorphic if and only if the spaces $C(K_1)$ and $C(K_2)$ are isometrically isomorphic.*

D. Amir [1] and M. Cambern [1] have strengthened this result as follows: *If there is an isomorphism T of $C(K_1)$ onto $C(K_2)$ such that $\|T\| \cdot \|T^{-1}\| < 2$, then K_1 and K_2 are homeomorphic.* The constant 2 is the best possible; there are compact metric spaces K_1 and K_2 such that $d(C(K_1), C(K_2)) = 2$ (H. B. Cohen [1]). However, if K_1 and K_2 are countable compacta, then $d(C(K_1), C(K_2)) \geqslant 3$ (Y. Gordon [1]).

An isometric classification of Lindenstrauss spaces is not known. Many interesting partial results can be found in Lindenstrauss and Wulbert [1] and Lazar and Lindenstrauss [1]. Let us note that the space c_0 is minimal among Lindenstrauss spaces in the following sense.

10.18 (Zippin [1]). *Every infinite-dimensional Lindenstrauss space X contains a subspace V which is isometrically isomorphic to the space c_0. Moreover, if X is separable, then the subspace V can be chosen so as to be the image of a contractive projection in the space X.*

The class of separable Lindenstrauss spaces admits a maximal member. More precisely:

10.19 (Pełczyński and Wojtaszczyk [1]). *There exists a separable Lindenstrauss space Γ with the property that for every separable Lindenstrauss space X and for every $\varepsilon > 0$ there is an isometrically isomorphic embedding T: $X \to \Gamma$ with $\|x\| \leqslant \|Tx\| \leqslant (1+\varepsilon)\|x\|$ for $x \in X$ and such that $T(X)$ is the image of a contractive projection from X.*

Wojtaszczyk [1] has shown that the space Γ with the above properties can be constructed in such a way that it is a Gurariĭ space of the universal arrangement (cf. Gurariĭ [1]), i.e. it has the following property:

($*$) *For every pair $F \supset E$ of finite-dimensional Banach spaces, for every isometrically isomorphic embedding T: $E \to \Gamma$ and for every $\varepsilon > 0$, there is an extension T: $F \to \Gamma$ such that $\|e\| \leqslant \|Te\| \leqslant (1+\varepsilon)\|e\|$ for $e \in E$.*

Gurariĭ [1] has shown that every Banach space satisfying condition ($*$) is a Lindenstrauss space and that the Gurariĭ space is unique up to an almost-isometry, i.e., if Γ_1 and Γ_2 are Gurariĭ spaces, then $d(\Gamma_1, \Gamma_2) = 1$. Luski [1] proved that the Gurariĭ space is isometrically unique.

The reader interested in the topics of this section is referred to the monograph by Lacey [1], which contains, among other things, proofs of the majority of the results stated here both for the real and for the complex scalars. Many results and an extensive bibliography on $C(K)$ spaces can be found in Semadeni's book [2]. For the connections of Lindenstrauss spaces with Choquet simplexes see Alfsen [1]. Further information can be found in the following surveys: Bernau–Lacey [1], Edwards [1], Lindenstrauss [2], [4], Proceedings of Conference in Swansea [1], and in the papers: Effross [1], [2], [3], Lazar [1], [2], [3], Lindenstrauss and Tzafriri [2].

§ 11. The isomorphic theory of $\mathscr{L}_p$ spaces

The isomorphic theory of $\mathscr{L}_p$ spaces is, in general, much more complicated than the metric theory of $L^p(\mu)$ spaces and Lindenstrauss spaces. The theory is still far from being completed. Many problems remain open. The only case in which the situation is clear is that of $p = 2$. From 9.7 it immediately follows

11.1. *A Banach space X is an $\mathscr{L}_2$ space if and only if it is isomorphic to a Hilbert space.*

The basic theorem of the general theory of $\mathscr{L}_p$ spaces is the following result, due to Lindenstrauss and Rosenthal [1]. (Recall that $p^* = p/(p-1)$ for $1 < p < \infty$; $p^* = 1$ for $p = \infty$; $p^* = \infty$ for $p = 1$.)

11.2. *Let $1 \leqslant p \leqslant \infty$ and $p \neq 2$. For every Banach space X which is not isomorphic to a Hilbert space the following statements are equivalent:*

(1) *X is an $\mathscr{L}_p$ space.*

(2) *There is a constant $c > 1$ such that, for every finite-dimensional subspace E of X, there are a finite-dimensional space l_n^p, a linear operator $T\colon l_n^p \to X$ and a projection P of X onto $T(l_n^p)$ such that $\|y\| \leqslant \|Ty\| \leqslant c\|y\|$ for $y \in l_n^p$, $T(l_n^p) \supset E$, $\|P\| \leqslant c$.*

(3) *X^* is isomorphic to a complemented subspace of a space $L^{p^*}(\mu)$.*

(4) *X^* is an $\mathscr{L}_{p^*}$ space.*

This yields the following corollary:

11.3. *We have*

(a) *Let $1 < p < \infty$ and let X be a Banach space which is not isomorphic to any Hilbert space. Then X is an $\mathscr{L}_p$ space if and only if X is isomorphic to a complemented subspace of a space $L^p(\mu)$.*

(b) *Every $\mathscr{L}_1$ space (resp. $\mathscr{L}_\infty$ space) is isomorphic to a subspace of an $L^1(\mu)$ space (resp. $L^\infty(\mu)$).*

(c) *If X is an $\mathscr{L}_1$ space (resp. an $\mathscr{L}_\infty$ space), then X^{**} is isomorphic to a complemented subspace of a space $L^1(\mu)$ (resp. $L^\infty(\mu)$).*

A Hilbert space can be isomorphically embedded as a complemented subspace of an $L^p(\mu)$ space for $1 < p < \infty$. (The subspace of L^p spanned by the Rademacher system $\{\operatorname{sgn}\sin 2^n\pi t\colon n = 0, 1, \ldots\}$ is such an example.) On the other hand, by Grothendieck [3], no complemented subspace of a space $L^1(\mu)$ is isomorphic to an infinite-dimensional Hilbert space. This is the reason why the assumption that X is not isomorphic to any Hilbert space does not appear in (b) and (c).

The paper Lindenstrauss and Rosenthal [1] contains many interesting characterizations of $\mathscr{L}_p$ spaces. Here we shall quote the following analogues of 10.13 and 10.14. Recall that a Banach space G is said to be *injective* if for every pair of Banach spaces $Z \supset Y$ and for every linear operator $T\colon Y \to G$, there is a linear operator $\tilde{T}\colon Z \to G$ which extends T.

11.4. *For every Banach space X the following statements are equivalent*:

(1) *X is an $\mathscr{L}_1$ space.*

(2) *For all Banach spaces Z and Y and any surjective linear operator $\Phi\colon Z \to Y$, every compact linear operator $T\colon X \to Y$ has a compact lifting $\tilde{T}\colon X \to Z$ (i.e. $T = \Phi\tilde{T}$).*

(3) *For all Banach spaces Z and Y and any surjective linear operator $\Phi\colon Z \to X$, every compact linear operator $T\colon Y \to X$ has a compact lifting $\tilde{T}\colon Y \to Z$.*

(4) *X^* is an injective Banach space.*

The reader interested in characterizations of $\mathscr{L}_p$ spaces in terms of Boolean algebras of projections (due to Lindenstrauss, Zippin and Tzafriri) is referred to Lindenstrauss and Tzafriri [2]. Other characterizations, in the language of operator ideals, can be found in Retherford and Stegall [1], Lewis and Stegall [1], in the surveys by Retherford [1] and Gordon, Lewis and Retherford [1] and in the monograph by Pietsch [1].

Now we shall discuss the problem of isomorphic classification of the spaces $\mathscr{L}_p$. If $1 < p < \infty$, then by 11.3, the problem reduces to that of isomorphic classification of complemented subspaces of spaces $L^p(\mu)$; also in the general case it is closely related to the latter problem. The latter problem is completely answered only for $l^p(S)$ spaces for $1 \leqslant p < \infty$. We have (Pełczyński [3], Köthe [2], Rosenthal [2]).

11.5. *Let $1 \leqslant p < \infty$. If X is a complemented subspace of a space $l^p(S)$ (resp. of $c_0(S)$), then X is isomorphic to a space $l^p(T)$ (resp. $c_0(T)$).*

To classify all separable $\mathscr{L}_p$ spaces for $1 < p < \infty$ one has to describe all complemented subspaces of L^p. This program is far of being completed. Lindenstrauss and Pełczyński [1] have observed that $L^p, l^p, l^p \times l^2$ and $E_p = (l^2 \times l^2 \times \ldots)_{l^p}$ are isomorphically distinct $\mathscr{L}_p$ spaces for $1 < p < \infty$, $p \neq 2$. Next Rosenthal [3], [4] has discovered less trivial examples of $\mathscr{L}_p$ spaces.

Let $\infty > p > 2$. Let X_p be the space of scalar sequences $x = (x(n))$

such that

$$\|x\| = \max\Big(\big(\sum_{n=1}^{\infty} |x(n)|^p\big)^{1/p}, \big(\sum_{n=1}^{\infty} |x(n)|^2/\log(n+1)\big)^{1/2}\Big) < \infty .$$

Let $B_p = (B_{p,1}\times B_{p,2}\times \ldots)_{l^p}$, where $B_{p,n}$ is the space of all square summable scalar sequences equipped with the norm

$$\|x\|_{B_{p,n}} = \max\Big(n^{1/p-1/2}\big(\sum_{j=1}^{\infty} |x(j)|^2\big)^{1/2}, \big(\sum_{j=1}^{\infty} |x(j)|^p\big)^{1/p}\Big).$$

For $1 < p < 2$ we put $X_p = (X_{p^*})^*$ and $B_p = (B_{p^*})^*$.

11.6 (Rosenthal). *Let $1 < p < \infty$, $p \neq 2$. The spaces X_p, B_p, $(X_p\times X_p\times\ldots)_{l^p}$, $X_p\times E_p$ and $X_p\times B_p$ are isomorphically distinct $\mathscr{L}_p$ spaces each different from L^p, l^p, $l^p\times l^2$, E_p.*

Taking "L_p-tensor powers" of X_p Schechtman [1] proved

11.7. *There exists infinitely many mutually non-isomorphic infinite-dimensional separable $\mathscr{L}_p$ spaces ($1 < p < \infty$, $p \neq 2$).*

Johnson and Odell [1] have proved

11.8. *If $1 < p < \infty$, then every infinite-dimensional separable $\mathscr{L}_p$ space which does not contain l^2 is isomorphic to l^p.*

11.8. yields the following earlier result of Johnson and Zippin [1].

11.9. *Let X be an infinite-dimensional $\mathscr{L}_p$ space with $1 < p < \infty$. If X is either a subspace or a quotient of l^p, then X is isomorphic to l^p.*

The above fact is also valid for the space c_0.

Now let us pass to $p = 1$. The problem of isomorphic classification of complemented subspaces of spaces $L^1(\mu)$ is a very particular case of that of isomorphic classification of $\mathscr{L}_1$ spaces. Even in the separable case neither of these problems is satisfactorily solved.

In contrast to 11.9 we have

11.10. *Among subspaces of l_1 there are infinitely many isomorphically distinct infinite dimensional $\mathscr{L}_1$ spaces.*

This has been established by Lindenstrauss [7]. His construction of the required subspaces $X_1, X_2, \ldots$ of l^1 is inductive and based on the fact that every separable Banach space is a linear image of l^1. $X_1 = \ker h_1$, where h_1 is a linear operator of l^1 onto L^1, and $X_{n+1} = \ker h_n$, where h_n is a linear operator of l^1 onto X_n for $n = 2, 3, \ldots$

We do not know whether the set of all isomorphic types of separable $\mathscr{L}_p$ spaces is countable ($1 \leqslant p < \infty, p \neq 2$).

In contrast to 11.10 the following conjecture is probable.

CONJECTURE. *Every infinite-dimensional complemented subspace of L^1 is isomorphic either to l^1 or to L^1.*

What we know is:

11.11 (Lewis and Stegall [1]). *If X is an infinite-dimensional complemented subspace of L^1 and X is isomorphic to a subspace of a separable dual space (in particular, to a subspace of l^1), then X is isomorphic to l^1.*

This implies that:

(a) *The space L^1 is not isomorphic to any subspace of a separable dual Banach space* (Gelfand [1], Pełczyński [2]).

(b) *The space l^1 is the only (up to isomorphisms) separable infinite-dimensional $\mathscr{L}_1$ space which is isomorphic to a dual space.*

The proof of (b) follows from 11.11, 11.3 (c) and the observation that every dual Banach space is complemented in its second dual.

In the non-separable case it is not known whether every dual $\mathscr{L}_1$ space is isomorphic to a space $L^1(\mu)$. Also it is not known which $L^1(\mu)$ spaces are isomorphic to dual spaces. For sigma-finite measures μ, $L^1(\mu)$ is isomorphic to a dual space iff μ is purely atomic (Pełczyński [2], Rosenthal [5]).

Now we shall discuss the situation for $p = \infty$. It seems to be the most complicated because of new phenomena which appear both in the separable and in the non-separable case. First, in contrast to the case of $1 \leqslant p < \infty$ (where there were only two isomorphic types of infinite-dimensional separable $L^p(\mu)$ spaces, namely L^p and l^p), there are infinitely many isomorphically different separable infinite-dimensional spaces $C(K)$. The complete isomorphic classification of such spaces is given in the next two theorems.

11.12 (Milutin [1]). *If K is an uncountable compact metric space, then the space $C(K)$ is isomorphic to the space C.*

For every countable compact space K, let $\alpha(K)$ denote the first ordinal α such that the αth derived set of K is empty.

11.13 (Bessaga and Pełczyński [2]). *Let K_1 and K_2 be countable infinite compact spaces such that $\alpha(K_1) \leqslant \alpha(K_2)$. Then the spaces $C(K_1)$ and $C(K_2)$ are isomorphic if and only if there is a positive integer n such that $\alpha(K_1) \leqslant \alpha(K_2) \leqslant \alpha(K_1)^n$.*

The theorem of Milutin 11.12 answers positively the question of Banach (cf. [B], p. 169).

It is easy to show that if K is a countable infinite compact space then the Banach space $(C(K))^*$ is isomorphic to l^1. Hence, by 11.13, there are uncountably many isomorphically different Banach spaces whose duals are isometrically isomorphic. This answers another question in [B], Rem. XI, § 9.

The problem of describing all isomorphic types of complemented subspaces of separable spaces $C(K)$ is open. The answer is known for c being isomorphic to c_0 (cf. 11.5) and $C(\omega^\omega)$ (Alspach [1]). This problem can be reduced to that of isomorphic classification of complemented subspaces of the space C. It is very likely that

CONJECTURE. *Every complemented subspace of C is isomorphic either to C or to C(K) for some countable compact metric space K.*

The following result of Rosenthal [6] strongly supports this conjecture.

11.14. *If X is a complemented subspace of C such that X^* is non-separable, then X is isomorphic to C.*

The class of isomorphic types of Lindenstrauss spaces is essentially bigger than that of complemented subspaces of $C(K)$. We have

11.15 (Benyamini and Lindenstrauss [1]). *There exists a Banach space BL with $(BL)^*$ isometrically isomorphic to l^1 and such that BL is not isomorphic to any complemented subspace of any space $C(K)$.*

From the construction of Benyamini and Lindenstrauss [1] it easily follows that, in fact, there are uncountably many isomorphically different spaces with the above property. Combining 11.15 with 10.19, we conclude that the Gurariĭ space Γ is also an example of a Lindenstrauss space which is not isomorphic to any complemented subspace of any $C(K)$.

Bourgain [1] gave a striking example of an infinite dimensional separable $\mathscr{L}_\infty$ space which does not have subspaces isomorphic to c_0; hence, by 10.18, it is not isomorphic to any Lindenstrauss space. Let us note that the results of Pełczyński [3] and Kadec and Pełczyński [1] imply

11.16. *If $1 \leqslant p < \infty$, then every infinite-dimensional $\mathscr{L}_p$ space has a complemented subspace isomorphic to l^p. Every infinite-dimensional complemented subspace of a space $C(K)$ contains isomorphically the space c_0.*

Our last result on separable $\mathscr{L}_\infty$ spaces is the following characterization of c_0.

11.17. *Every Banach space E isomorphic to c_0 has the following property:*

(S) *If F is a separable Banach space containing isometrically E, then E is complemented in F.*

Conversely, if an infinite-dimensional separable Banach space E has property (S), *then E is isomorphic to c_0.*

The first part of 11.17 is due to Sobczyk [1] (cf. Veech [1] for a simple proof). The second part is due to Zippin [3]. A particular case of Zippin's result, assuming that E is isomorphic to a $C(K)$ space, was earlier obtained by Amir [2].

Now we shall be concerned with the problem of isomorphic classification of non-separable spaces $C(K)$. The multitude of different non-separable spaces $C(K)$ and the variety of their isomorphical invariants is so rich that there is almost no hope of obtaining any complete description of the isomorphic types of non-separable spaces $C(K)$, even for K's of cardinality continuum. The results which have been obtained concern special classes of spaces $C(K)$ and their complemented subspaces. Among general conjectures the following seems to be very probable.

CONJECTURE. *Every $C(K)$ space is isomorphic to a space $C(K_0)$ for some compact totally disconnected Hausdorff space K_0.*

The following result is due to Ditor [1].

11.18. *For every compact Hausdorff space K, there exist a totally disconnected compact Hausdorff space K_0, a continuous surjection $\varphi: K \longrightarrow K_0$ and a contractive positive projection $P: C(K_0) \underset{\text{onto}}{\longrightarrow} \varphi^0(C(K))$, where $\varphi^0: C(K) \longrightarrow C(K_0)$ is the isometric embedding defined by $\varphi^0(f) = f \circ \varphi$ for $f \in C(K)$. Hence $C(K)$ is isometric to a complemented subspace of $C(K_0)$.*

An analogous result for compact metric spaces was earlier established by Milutin [1], cf. Pełczyński [4].

The theorem of Milutin 11.12 can be generalized only to special classes of non-metrizable compact spaces. Recall that the *topological weight* of a topological space K is the smallest cardinal n such that there exists a base of open subsets of K of cardinality n. We have (Pełczyński [4])

11.19. *Let K be a compact Hausdorff space whose topological weight is an infinite cardinal n. If K is either a topological group or a product of a family of metric spaces, then $C(K)$ is isomorphic to $C([0, 1]^n)$.*

In particular, for every compact space K satisfying the assumptions of 11.19, the space $C(K)$ is isomorphic to its Cartesian square. This property is not shared by arbitrary infinite compact Hausdorff spaces. We have (Semadeni [1])

11.20. *Let ω_1 be the first uncountable ordinal and let $[\omega_1]$ be the space of all ordinals which are $\leqslant \omega_1$ with the natural topology determined by the order. Then the space $C([\omega_1])$ is not isomorphic to its Cartesian square.*

Numerous mathematicians have studied injective spaces (whose definition was given before 11.4). Theorem 10.15 of Nachbin, Goodner and Kelley suggests the following

CONJECTURE. *Every injective Banach space is isomorphic to a space $C(K)$ for some extremally disconnected compact Hausdorff space K.*

It is easy to see that: (1) every complemented subspace of an injective space is injective, (2) every space $l^\infty(S)$ is injective, (3) a Banach space is injective if and only if it is complemented in every Banach space containing it isometrically, (4) every Banach space X is isometrically isomorphic to a subspace of the space $l^\infty(S)$, where S is the unit sphere of X^*. From the above remarks it follows that

11.21. *A Banach space X is injective if and only if it is isometrically isomorphic to a complemented subspace of a space $l^\infty(S)$.*

Lindenstrauss [3] has shown (cf. 11.5):

11.22. *Every infinite-dimensional complemented subspace of l^∞ $(= l^\infty(S)$ for a countable infinite $S)$ is isomorphic to l^∞.*

As a corollary from this theorem we get the following earlier result of Grothendieck [3].

11.23. *Every separable injective Banach space is finite-dimensional.*

Theorem 11.22 cannot be generalized to the spaces $l^\infty(S)$ with uncountable S. In fact, we have

11.24 (Akilov [1]). *For every measure μ the space $L^\infty(\mu)$ is injective.*

11.25 (Pełczyński [3], [5], Rosenthal [5]). *Let μ be a sigma-finite measure. Then the space $L^\infty(\mu)$ is isomorphic to $l^\infty(S)$ if and only if the measure μ is separable (i.e. the space $L^1(\mu)$ is separable).*

Theorem 11.24 is closely related to the following

11.26. (a) *An $\mathscr{L}_\infty$ space isomorphic to a dual space is injective.*

(b) *An injective bidual space is isomorphic to an $L^\infty(\mu)$.*

11.26 (a) follows from 11.4 (4) because by Diximier [1] every dual Banach space is complemented in its second dual. 11.26 (b) is due to Haydon [1].

Applying deep results of Solovey and Gaifman concerning complete Boolean algebras, Rosenthal [5] has shown that

11.27. *There exists an injective Banach space which is not isomorphic to any dual Banach space.*

Let us mention that Isbell and Semadeni [1] have proved that

11.28. *There exists a compact Hausdorff space K which is not extremally disconnected and is such that $C(K)$ is injective.*

Concluding this section, let us notice that the "dual problem" to the last conjecture is completely solved. Namely (cf. 10.16) we have

11.29 (Köthe [2]). *For every Banach space X the following statements are equivalent:*

(1) *X is projective, i.e. for every pair E, F of Banach spaces, for every linear surjection $h\colon F\to E$ and for every linear operator $T\colon X\to E$, there exists a linear operator $\tilde{T}\colon X\to F$ which lifts T, i.e. $h\tilde{T} = T$.*

(2) *X is isomorphic to a space $l^1(S)$.*

The reader interested in the problems discussed in this section is referred to Lindenstrauss and Tzafriri [1], [2], Semadeni [2], Bade [1], Pełczyński [4] and Ditor [1], Lindenstrauss [2], [4], Rosenthal [9], and to the references in the above mentioned books and papers, see also "Added in proof".

§ 12. The isomorphic structure of the spaces $L^p(\mu)$

The starting point for the discussion of this section is [B], Chap. XII. We shall discuss the following question:

I. Given $1 \leqslant p_1 < p_2 < \infty$. What are the Banach spaces E which are simultaneously isomorphic to a subspace of L^{p_1} and to a subspace of L^{p_2}?

One can ask more generally:

II. Which Banach spaces X are isomorphic to subspaces of a given space $L^p(\mu)$?

One of the basic results in this direction is theorem 3.2 of this survey, which can be restated as follows:

12.1. *A Banach space E is (isometric) isomorphic to a subspace of a space $L^p(\mu)$ iff E is locally (isometrically) isomorphically representable in l^p.*

We shall restrict our discussion to the case where $1 \leqslant p < \infty$ and E is a separable Banach space. Since every separable subspace of the space $L^p(\mu)$ is isometrically isomorphic to a subspace of L^p, in the sequel we shall study isomorphic properties of the spaces L^p. It turns out that the case $2 < p < \infty$ is much simpler than that of $1 \leqslant p < 2$. The following concepts will be useful in our discussion.

Definition. Let $1 \leqslant p < \infty$. We shall say that a subspace E of the space L^p is a *standard image of* l^p if there exist isomorphisms T: $l^p \xrightarrow[\text{onto}]{} E$ and U: $L^p \xrightarrow[\text{onto}]{} L^p$ such that, for $n \neq m$ $(n, m = 1, 2, \ldots)$, the intersections of the supports of the functions $UT(e_n)$ and $UT(e_m)$ have measure zero. Here e_n (for $n = 1, 2, \ldots$) denotes the nth unit vector in the space l^p.

A subspace E of the space L^p will be called *stable* if it is closed in the topology of the convergence in measure, i.e. for every sequence (f_n) of elements of E, the condition $\lim_n \int_0^1 |f_n(t)|/(1+|f_n(t)|)\,dt = 0$ implies $\lim_n \|f_n\|_p = 0$.

It is easy to see that

12.2. (a) *Every sequence of functions in L^p which have pair-wise disjoint supports spans a standard image of l^p.*

(b) *Every standard image of l^p is complemented in L^p.*

Much deeper, especially for $1 \leqslant p < 2$, is the next result, which shows that the property of subspaces of L^p of being stable does not depend on the location of the subspace in the space.

12.3. *Let $1 \leqslant p < \infty$ and $p \neq 2$. Then, for every infinite-dimensional subspace E of the space L^p, the following statements are equivalent*:

(1) *E is stable.*

(2) *No subspace of E is a standard image of l^p.*

(3) *No subspace of E is isomorphic to l^p.*

Moreover, if $p > 1$, conditions (1)–(3) are equivalent to those stated below:

(4) *There exists a $q \in [1, p)$ and a constant C_q such that*

$$(*) \qquad \|f\|_p \leqslant \|f\|_q \leqslant C_q \|f\|_p \quad \text{for} \quad f \in E.$$

(5) *For every $q \in [1, p)$ there is a C_q such that* (*) *holds.*

The last theorem, for $p > 2$, is due to Kadec and Pełczyński [1], and for $1 \leqslant p < 2$, is due to Rosenthal [7]. The following result of Kadec and Pełczyński [1] is an immediate corollary of 12.3.

12.4. *Let E be an infinite-dimensional subspace of a space L^p with $2 < p < \infty$. Then E is stable if and only if E is isomorphic to a Hilbert space.*

Suppose that $2 < p < \infty$ and E is a subspace of L^p which is isomorphic to a Hilbert space. Then, by 12.4 and by the condition 12.3 (5) with $q = 2$, the orthogonal (with respect to the L^2 inner product) projection of L^p onto E is continuous as an operator from L^p into L^p. Hence, by 12.3 (2) and 12.2 (b), we get

12.5. *Let $2 < p < \infty$ and let E be a subspace of L^p. Then*:

(a) *if E is isomorphic to a Hilbert space, then E is complemented in L^p*;

(b) *if E is not isomorphic to any Hilbert space, then E contains a complemented subspace isomorphic to l^p.*

The next result is due to Johnson and Odell [1].

12.6. *Suppose that E is a subspace of a space L^p with $2 < p < \infty$. Then E is isomorphic to a subspace of the space l^p if and only if no subspace of E is isomorphic to a Hilbert space.*

The assumption of 12.6 that $p > 2$ is essential. For each p with $1 \leqslant p < 2$, there is a subspace E of L^p such that E is not isomorphic to any subspace of l^p and no infinite dimensional subspace of E is stable (Johnson and Odell [1]).

Now we shall discuss the situation for $1 \leqslant p < 2$. In this case there are many isomorphically different stable subspaces of the space L^p. The crucial fact is the following theorem, which goes back to P. Levy [1]; however, it was stated in the Banach space language much later (by Kadec [4] for l^q, and by Bretagnolle, Dacunha-Castelle and Krivine [1] and Lindenstrauss and Pełczyński [1] in the general case).

12.7. *If $1 \leqslant p < q \leqslant 2$, then the space L^p contains a subspace E_q isometrically isomorphic to L^q.*

The proof of 12.7 employs a probabilistic technique. Its idea is the following:

1. For every q with $1 < q \leqslant 2$, there exists a random variable (= measurable function) ξ_q: $R \to R$ which has the characteristic function

$$\hat{\xi}_q(s) = \int_R \exp(\xi_q(t) \cdot is)\, dt = \exp(-|s|^q)$$

and is such that, for each $p < q$, $\xi_q \in L^p(R)$. By $L^p(R^n)$ we denote here the space $L^p(\lambda)$, where λ is the n-dimensional Lebesgue measure for R^n.

2. Let $\xi_{q1}, \ldots, \xi_{qn}$ be independent random variables each of the same

distribution as ξ_q, for instance let $\xi_{qj} \in L^p(R^n)$ be defined by $\xi_{qj}(t_1, t_2, \ldots, t_n) = \xi_q(t_j)$. Assume that $c_1, \ldots, c_n$ are real numbers such that $\sum_{j=1}^{n} |c_j|^q = 1$, and let $\eta = \sum_{j=1}^{n} c_j \xi_{qj}$. Since the random variables $\xi_{q1}, \ldots, \xi_{qn}$ are independent and have the same distribution and hence the same characteristic functions as ξ_q, we have

$$\hat{\eta}(s) = \sum_{j=1}^{n} c_j \hat{\xi}_{qj}(s) = \sum_{j=1}^{n} \exp(-|sc_j|^q)$$
$$= \exp\left(-|s|^q \cdot \sum_{j=1}^{n} |c_j|^q\right) = \exp(-|s|^q) = \hat{\xi}_q(s).$$

Hence η has the same distribution as ξ_q and therefore

$$(*) \qquad \Big\| \sum_{j=1}^{n} c_j \xi_{pj} \Big\|_p = \|\eta\|_p = \|\xi_q\|_p \quad \text{if} \quad \sum_{j=1}^{n} |c_j|^q = 1,$$

for every p with $1 \leqslant p < q$.

3. By $(*)$, the linear operator $T\colon l_n^p \to L^p(R^n)$ defined by $T(c_1, \ldots, c_n) = \|\xi_q\|_p^{-1} \cdot \sum_{j=1}^{n} c_j \xi_{qj}$ is an isometric embedding. Hence L^q is locally representable in l^p. Applying 12.1 we complete the proof.

By Banach [B], p. 186, Théorème 10, and the fact that the space l^1 is not reflexive, it follows that if $1 \leqslant p < q < 2$, then l^p is not isomorphic to any subspace of L^q. Hence, by 12.3, the subspaces E_q of 12.7 are stable.

Theorem 12.7 can be generalized as follows (Maurey [1]):

12.8. *Let $1 < p \leqslant q < 2$. Then, for every measure μ, there exists a measure ν such that the space $L^q(\mu)$ is isometrically isomorphic to a subspace of the space $L^p(\nu)$.*

Rosenthal [7] has discovered another property of stable subspaces of L^p, which can be called the extrapolation property.

12.9. *If $1 \leqslant p < \infty, p \neq 2$, and E is a stable subspace of the space L^p, then there exist an isomorphism U of L^p onto itself and an $\varepsilon > 0$ such that $U(E)$ is a closed stable subspace of the space $L^{p+\varepsilon}$,* i.e. *there is a $C > 0$ such that $\|f\|_p \leqslant \|f\|_{p+\varepsilon} \leqslant C\|f\|_p$ for every $f \in E$.*

Combining 12.9 with the result of Kadec and Pełczyński [1] showing that

12.10. *Every non-reflexive subspace of L^1 contains a standard image of l^1,*

we obtain the following:

12.11 (Rosenthal [7]). *Every reflexive subspace of the space L^1 is stable, hence isomorphic to a subspace of a space L^p for some $p > 1$.*

The results of Chap. XII of [B] and Orlicz [2], Satz 2 combined with

12.3, 12.4 and 12.7 yield an answer to question (I) stated at the beginning of this section and to the question in [B] on p. 186. We have

12.12. *Let E be an infinite-dimensional Banach space and let* $1 \leqslant p < q < \infty$. *E is isomorphic to a subspace of* L^p *and to a subspace of* L^q *if and only if E is isomorphic to a subspace of* $L^{\min(q,2)}$. *In particular, if* $q \leqslant 2$, *then* $\dim_l L^p \geqslant \dim_l L^q \geqslant \dim_l l^q$, *and if* $p \neq 2 < q$, *then* $\dim_l L^p$ *is incomparable with* $\dim_l L^q$ *and with* $\dim_l l^q$.

The fact that, for $2 < p < q$, the linear dimensions of L^p and l^q are incomparable has been established first by Paley [1]. The incomparability of $\dim_l L^p$ and $\dim_l L^q$ for $q > 2 > p$ is due to Orlicz [2]. For $1 < p < \infty$, $p \neq 2$, there exist the subspaces of the space L_p which are isomorphic to l^p but are not standard images of l^p. This is a consequence of the following theorem of Rosenthal [3], [8], and Bennett, Dor, Goodman, Johnson and Newman [1].

12.13. *If either* $1 < p < \infty$, $p \neq 2$, *then there exists a non-complemented subspace of* l^p *which is isomorphic to the whole space.*

It is not known whether every subspace of l^1 which is isomorphic to l^1 is complemented in the whole space.

By 12.7 and the fact that, for $p \neq q$ no subspace of l^p is isomorphic to l^q, it follows that the assumption $p > 2$ in 12.5 (b) is indispensable. The following result is related to 12.5 (a):

12.14. (a) *Let* $1 < p < 2$ *and let E be an infinite-dimensional subspace of the space* L^p. *If E is isomorphic to the Hilbert space, then E contains an infinite-dimensional subspace which is complemented in* L^p.

(b) *If* $1 \leqslant p < \infty$, $p \neq 2$, *then there exists a non-complemented subspace of* L^p *which is isomorphic to a Hilbert space.*

Part (a) is due to Pełczyński and Rosenthal [1], and part (b) – to Rosenthal [8] for $1 \leqslant p \leqslant 4/3$ and to Bennett, Dor, Goodman, Johnson and Newman for all p with $1 \leqslant p < 2$.

In connection with the table in [B], p. 215 (property (15)) let us observe (cf. Pełczyński [3] and 5.2) that

12.15. *If* $1 \leqslant p < \infty$, $p \neq 2$, *then there exists an infinite-dimensional closed linear subspace of* l^p *which is not isomorphic to the whole space.*

The following theorem of Johnson and Zippin [1] gives a description of subspaces with the approximation property of the spaces l^p.

12.16. *If E is a subspace of a space* l^p *with* $1 < p < \infty$, *and E has the approximation property, then E is isomorphic to a complemented subspace of a product space* $(G_1 \times G_2 \times \ldots)_{l^p}$, *where* G_n*'s are finite-dimensional subspaces of the space* l^p.

CHAPTER VI

§ 13. The topological structure of linear metric spaces

The content of [B], Rem. XI, § 4 was a catalyst for intensive investigations of the topological structure of linear metric spaces and their subsets. These investigations have lead to the following theorem.

13.1. ANDERSON-KADEC THEOREM. *Every infinite-dimensional, separable, locally convex complete linear metric space is homeomorphic to the Hilbert space* l^2.

This result fully answers one of the questions raised in [B], Rem. XI, § 4, p. 212 and disproves the statement that the space s is not homeomorphic to any Banach space ([B], Rem. IV, § 1, p. 206). Theorem 13.1 is a product of combined efforts of Kadec [11], [12], Anderson [1] and Bessaga and Pełczyński [5], [6]. For alternative or modified proofs see Bessaga and Pełczyński [7] and Anderson and Bing [1]. Earlier partial results can be found in papers by Mazur [1], Kadec [6], [7], [8], [9], [10], Kadec and Levin [1], Klee [1], Bessaga [1].

In the proofs of 13.1 and other results on homeomorphisms of linear metric spaces three techniques are employed:

A. *Kadec's coordinate approach.* The homeomorphism between spaces X and Y is established by setting into correspondence the points $x \in X$ and $y \in Y$ which have the same "coordinates". The "coordinates" are defined in metric terms with respect to suitably chosen uniformly convex norms (see the text after 1.9 for the definition) of the spaces.

B. *The decomposition method*, which consists in representing the spaces in question as infinite products, and performing on the products suitable "algebraic computations" originated by Borsuk [1] (cf. [B], Chap. XI, § 7, Théorèmes 6–8). For the purpose of stating some results, we recall the definition of topological factors. Let X and Y be topological spaces. Y is said to be a *factor of* X (written $Y|X$) if there is a space W such that X is homeomorphic to $Y \times W$. A typical result obtained with the use of the decomposition method is the following criterion, due to Bessaga and Pełczyński [5], [6]:

13.2. *Let X and H be a Banach space and an infinite-dimensional Hilbert space, respectively, both of the same topological weight. Then $H|X$ implies that X is homeomorphic to H.*

Many applications of 13.2 depend on the following result of Bartle and Graves [1] (see also Michael [1], [2], [3] for a simple proof and generalizations).

13.3. *Let X be a Banach space. If Y is either a closed linear subspace or a quotient space of X, then $Y|X$.*

Notice that both 13.2 and 13.3 are valid under the assumption that X is merely a locally convex complete linear metric space.

Also the next result due to Toruńczyk [3], [4], [5], and some of its generalizations give rise to applications of the decomposition method.

13.4. *If X is a Banach space and A is an absolute retract for metric spaces which can be topologically embedded as a closed subset of X, then $A|(X \times X \times \ldots)_{l^2}$. If H is an infinite-dimensional Hilbert space and A is a complete absolute retract for metric spaces and the topological weight of A is less than or equal to that of H, then $A|H$.*

C. *The absorption technique*, which gives an abstract framework for establishing homeomorphisms between certain pairs (X, E) and (Y, F) consisting of metric spaces and their subsets, when X and Y are already known to be homeomorphic. (The pairs (X, E) and (Y, F) are said to be *homeomorphic*, in symbols $(X, E) \sim (Y, F)$, if there is a homeomorphism h of X onto Y which carries E onto F, and hence carries $X \backslash E$ onto $X \backslash F$). A particular model designed for identifying concrete spaces homeomorphic to R^∞ can briefly be described as follows. Consider the Hilbert cube $Q = [-1, 1]^\infty$ and its pseudo-interior $P = (-1, 1)^\infty$, which is obviously homeomorphic to R^∞. It turns out that every subset $A \subset Q$ which is such that $(Q, A) \sim (Q, Q \backslash P)$ can be characterized by certain property involving extensions and approximations of maps and related to Anderson's [2] theory of Z-sets, called cap (for compact absorption property). Hence, in order to show that a metric space E is homeomorphic to R^∞ it is enough to represent E as a subset of a space X homeomorphic to Q so that the complement $X \backslash E$ has cap. For applying this technique it is convenient to have many models for the Hilbert cube. An important role in this respect is played by the following classical theorem, due to Keller [1],

13.5. *Every infinite-dimensional compact convex subset of the Hilbert space l^2 is homeomorphic to the Hilbert cube,*

and the remark of Klee [4]

13.6. *Every compact convex subset of any locally convex linear metric space is affinely embeddable into l^2.*

For more details concerning the model presented here and other models of the absorption technique see papers by Anderson [4], Bessaga and Pełczyński [8], [7], [9], Toruńczyk [2] and the book by Bessaga and Pełczyński [10], Chapters IV, V, VI, VIII. The most general axiomatic setting for "absorption" with miscellaneous applications is presented by Toruńczyk [2] and Geoghegan and Summerhill [1].

During the years 1966–1977 several authors attempted to extend the Kadec Anderson theorem to Banach spaces of an arbitrary topological weight; for the information see Bessaga and Pełczyński [1], Chap. VII, and also Toruńczyk [5], Terry [1]. The final solution has been obtained only recently by Toruńczyk [6] who proved

13.7. *Let X be a complete metric space which is an absolute retract for metric spaces and let $\aleph = wX$, the density character of X. Then X is homeomorphic to the Hilbert space $l_2(\aleph)$ if and only if the following two conditions are satisfied:*

(a) *$X \times l_2$ is homeomorphic to X,*

(b) *every closed subset A of X with $wA < \aleph$ is a Z-set, i.e. for every compact $K \subset X$ the identity embedding of K into X is the uniform limit of a sequence of continuous maps of K into $X \backslash A$.*

In particular,

13.8. *Every locally convex complete metric linear space is homeomorphic to a Hilbert space.*

Detailed proofs and other characterizations of Hilbert spaces and Hilbert spece manifolds can be found in Toruńczyk [6].

It is natural to ask if in the Anderson–Kadec Theorem 13.1 the assumption of local convexity is essential. The problem is open and only very special non-locally convex spaces are known to be homeomorphic to l_2. For instance (Bessaga and Pełczyński [9]):

13.9. *The space S* ([B], Introduction, § 7, p. 30) *is homeomorphic to l^2. More generally, if X is a separable complete metric space which has at least two different points, then the space M_X of all Borel measurable maps f: $[0, 1] \to X$ with the topology of convergence in* (*the Lebesgue*) *measure is homeomorphic to l^2.*

More examples are presented in Bessaga and Pełczyński [10], Chap. VI.

It is known that a non-complete normed linear space cannot be homeomorphic to any Banach space. This easily follows from the theorem of Mazur and Sternbach [1] that every G_δ linear subspace of a Banach space must be closed. There are at least $\aleph_1$ topologically different separable normed linear spaces which can be distinguished by their absolute Borel types (Klee [5], and Mazur – unpublished). Henderson and Pełczyński have proved that even among sigma-compact normed linear spaces there are at least $\aleph_1$ topologically different (cf. Bessaga and Pełczyński [10], Chapter VIII, § 5).

It is not known whether every normed linear space is homeomorphic to an inner product space.

Using suitable absorption models, one can prove (Bessaga and Pełczyński [8] and [10], Chap. VIII, § 5, Toruńczyk [2])

13.10. *If X is an infinite-dimensional normed linear space which is a countable union of its finite-dimensional compact subsets, then X is homeomorphic to the subspace $\sum R$ of R^{∞} consisting of all sequences having at most finitely many non-zero coordinates. If X is a sigma-compact normed linear space containing an infinite-dimensional compact convex subset, then X is homeomorphic to the pseudo-boundary $Q\backslash P$ of the Hilbert cube.*

For more details on topological classification of non-complete linear metric spaces the reader is referred to Bessaga and Pełczyński [10], Chap. VIII and the references therein.

Another interesting problem is to find which subsets of a given infinite-dimensional Banach space are homeomorphic to the whole space. The situation is completely different from that in the finite-dimensional case. For instance, we have

13.11. *Let X be an infinite-dimensional Banach space. Then the following kinds of subsets X are homeomorphic to the whole space*:

(i) *spheres,*

(ii) *arbitrary closed convex bodies* (= *closed convex sets with non empty interior*), *in particular*: *closed balls, closed half-spaces, strips between two half-spaces and so on,*

(iii) *the sets $X\backslash A$, where A is sigma-compact.*

This result for the space l^2 and several other special spaces has been obtained by Klee [3], [6]. The general case can be reduced to that of l^2 by factoring from X a separable space, homeomorphic to l^2, and by applying some additional constructions, cf. Bessaga and Pełczyński [10], Chap. VI.

The investigations of topological structure of linear metric spaces resulted in active development of the theory of infinite-dimensional manifolds. If E is a linear metric space, then by a *topological manifold modelled on E* (briefly: an *E-manifold*) we mean a metrizable topological space M which has an open cover by sets homeomorphic to open subsets of E. In the same manner one defines manifolds modelled on the Hilbert cube.

A fundamental theorem on topological classification of manifolds with a fixed model E, an infinite-dimensional linear metric space satisfying certain conditions, is due to Henderson (see Henderson [1], [2] and Henderson and Schori [1]). For simplicity we state this theorem in the case of Hilbert spaces.

13.12. *Let H be an infinite-dimensional Hilbert space. Then every connected*

H-manifold is homeomorphic to an open subset of H. H-manifolds M_1 and M_2 are homeomorphic if and only if they are of the same homotopy type, i.e. *there are continuous maps $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_1$ such that the compositions gf and fg are homotopic to the identities id_{M_1} and id_{M_2}, respectively.*

For analogous results on infinite-dimensional differential manifolds, see Burghelea and Kuiper [1], Eells and Elworthy [1], Elworthy [1], Moulis [1].

The systematic theory of manifolds modelled on the Hilbert cube has been developed by Chapman [2], [3], [4], [5] and is closely related to the simple homotopy theory of polyhedra (Chapman [5], [6], cf. Appendix to Cohen [1]) and has some points in common with Borsuk's shape theory (Chapman [1]). Chapman [7] is an excellent source of information.

We conclude this section with some comments concerning the classification of Banach spaces with respect to uniform homeomorphisms. Banach spaces X and Y are *uniformly homeomorphic* if there exists a homeomorphism $f\colon X \xrightarrow[\text{onto}]{} Y$ such that both f and f^{-1} are uniformly continuous.

There are non isomorphic but uniformly homeomorphic Banach spaces (Aharoni and Lindenstrauss [1]). However, Enflo [1] has proved that a Banach space which is uniformly homeomorphic to a Hilbert space is already isomorphic to the Hilbert space (cf. 9.13 here).

Combining the results of Lindenstrauss [10] and Enflo [5] we get

13.13. *If $1 \leqslant p < q \leqslant \infty$, then, for arbitrary measures μ and ν, the spaces $L^p(\mu)$ and $L^q(\nu)$ are not uniformly homeomorphic, except the case where* $\dim L^p(\mu) = \dim L^q(\nu) < \infty$.

To state the next result (due to Lindenstrauss [10]) we recall that a closed subspace S of a metric space M is said to be a *uniform retract of M* if there is a uniformly continuous map $r\colon M \to S$ such that $r(x) = x$ for $x \in S$.

13.14. *If a linear subspace Y of a Banach space X is a uniform retract of X and $\varkappa(Y)$ is complemented in Y^{**}, then Y is complemented in X.*

Observe that if Y is reflexive or, more generally, conjugate to a Banach space, then $\varkappa(Y)$ is complemented in Y^{**} (cf. Diximier [1]).

On the other hand, we have (see Lindenstrauss [10])

13.15. *Let K be a compact metric space. Then every isometric image of $C(K)$ in an arbitrary metric space M is a uniform retract of M.*

Combining 13.14 and 13.15 with the result of Grothendieck [3] (cf. Pełczyński [3]) that no separable infinite-dimensional conjugate Banach space is complemented in a $C(K)$, we get

13.16. *If K is an infinite compact metric space, then the space $C(K)$ is not uniformly homeomorphic to any conjugate Banach space.*

Enflo [6] has shown that

13.17. *No subset of a Hilbert space is uniformly homeomorphic to the space C.*

In "Added in proof" we present Aharoni's and Ribe's contributions to the classification of Banach spaces with respect to uniform homeomorphisms.

Uniform homeomorphisms of locally convex complete metric spaces have been studied by Mankiewicz [1], [2], cf. also Bessaga [1], § 11. In particular, Mankiewicz [2] has proved that

13.18. *If X is one of the spaces $l^2, s, l^2 \times s$ and Y is a locally convex linear metric space which is uniformly homeomorphic to X, then Y is isomorphic to X.*

From 13.18 it immediately follows that s is not uniformly homeomorphic to l_2 (a more general fact is proved in Bessaga [1], p. 282).

§ 14. Added in proof

Ad § 2. The following basic fact in the isomorphic theory of Banach spaces, due to H. P. Rosenthal, is related to the discussion in § 9 Chap. IX and to Example 2 in § 3 of this survey.

14.1. *Let (x_n) be a bounded sequence in a Banach space. Then (x_n) contains a subsequence equivalent to the standard vector basis of l^1 iff (x_n) has a subsequence whose no subsequence is a weak Cauchy sequence.*

For the proof (for real Banach spaces) see Rosenthal [11]; Dor [1] has adjusted Rosenthal's proof to cover the complex spaces. For related but more delicate results the reader is referred to the excellent survey by Rosenthal [12] and to the papers: Odell and Rosenthal [1] and Bourgain, Fremlin and Talagrand [1].

For further information on *WCG* spaces and renorming problems the reader is referred to the lecture notes by Diestel [1] and to the book by Diestel and Uhl [1].

Ad § 3. Theorems 13.7 and 13.8 generalize to the case of arbitrary $p \in (1, \infty)$. We have

14.2 (Krivine [2]). *Let $1 < p < \infty$. Then l^p is locally representable in a Banach space X iff l^p is locally a-representable in X for some $a \geqslant 1$.*

For an alternative proof of 14.2 see Rosenthal [10].

Using 14.2, Maurey and Pisier [3] have established

14.3. *Let X be a Banach space, let p_X (resp. q_X) be the supremum (resp. infimum) of $p \in [1, \infty]$ such that there is a positive $C = C(q, X) < \infty$ with* *the property that, for every finite sequence* (x_j) *of elements*

$$\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|\,dt \leqslant C\big(\sum_j \|x_j\|^q\big)^{1/q}$$

$$\Big(\textit{resp.}\ \int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|\,dt \geqslant C\big(\sum_j \|x_j\|^q\big)^{1/q}\Big),$$

where (r_j) *are the Rademacher functions.*

Then l^{p_X} *and* l^{q_X} *are locally representable in* X.

Observe that $1 \leqslant p_X \leqslant 2$ and $\infty \geqslant q_X \geqslant 2$. (The right-hand side inequalities follow from Dvoretzky's Theorem.) In the limit case $p_X = 1$ (resp. $q_X = \infty$) Theorem 14.3 yields 13.8 equivalence (i) and (iv) (resp. 13.7).

Entirely different criterion of local representability of l^1 was discovered by Milman and Wolfson [1].

14.4. *Let* X *be an infinite-dimensional Banach space with the property that there is a* $C < \infty$ *such that for every* $n = 1, 2, \ldots$ *there is an n-dimensional subspace, say* E_n*, of* X *with* $d(E_n, l_n^2) \leqslant C\sqrt{n}$*. Then* l^1 *is locally representable in* X.

Ad § 4. R. C. James [14] improved 4.3 by constructing a non-reflexive Banach space of type 2, i.e. satisfying 13.8 (iv) with $q = 2$.

The reader interested in the subject discussed in § 4 is refered to the books and notes: Lindenstrauss and Tzafriri [1], volume II, Maurey and Schwartz [1] (various exposés by Maurey, Maurey and Pisier, and Pisier), Diestel [1], and to the papers: Figiel [6], [7], [8], and Pisier [2].

Ad § 5.

14.5 (Szankowski [4]). *The space of all bounded linear operators from* l^2 *into itself fails to have the approximation property.*

Ad § 8. The following result, due to Maurey and Rosenthal [1], is related to the question whether every infinite-dimensional Banach space contains an infinite-dimensional subspace with an unconditional basis.

14.6. *There exists a Banach space which contains a weakly convergent to zero sequence of vectors of norm one such that no infinite subsequence of the sequence forms an unconditional basis for the subspace which it spans.*

Ad § 9. The paper by Enflo, Lindenstrauss and Pisier [1] contains an example of a Banach space X which is not isomorphic to a Hilbert space but which has a subspace, say Y, such that both Y and X/Y are isometrically isomorphic to l^2 (cf. also Kalton and Peck [1]).

Ad §§ 10 and 11. We recommend to the reader the surveys: Rosenthal [9], [12]. The reader might also consult the book by Diestel and Uhl [1].

Most of the recent works on $C(K)$ spaces concern non-separable $C(K)$ spaces. The reader is referred to Alspach and Benyamini [1], Argyros and Negropontis [1], Benyamini [2], Dashiell [1], Dashiell and Lindenstrauss [1], Ditor and Haydon [1], Etcheberry [1], Hagler [1], [2], Haydon [1], [2], [3], [4], Gulko and Oskin [1], Kislyakov [1], Talagrand [1], Wolfe [1]. The separable $C(K)$ spaces are studied in the papers: Alspach [1], Benyamini [1], Billard [1], Zippin [1].

Ad § 12. The reader interested in the subject should consult the seminar notes by Maurey and Schwartz [1] and the memoir by Johnson, Maurey, Schechtman and Tzafriri [1]. The reader is also referred to the survey by Rosenthal [9] and to the papers: Alspach, Enflo and Odell [1], Enflo and Rosenthal [1], Enflo and Starbird [1], Gamlen and Gaudet [1], Stegall [1], [2].

Ad § 13. The following result of Ribe [1] shows that, despite the example of Aharoni and Lindenstrauss [1] mentioned in § 13, the classification of Banach spaces with respect to uniform homeomorphisms is "close" to linear topological classification.

14.7. *If Banach spaces X and Y are uniformly homeomorphic, then there is an $a \geqslant 1$ such that X is locally a-representable in Y and Y is locally a-representable in X.*

It is known, however (Enflo oral communication), that the spaces L^1 and l^1, which are obviously locally representable each into the other, are not uniformly homeomorphic. On the other hand, isomorphically different Banach spaces might have the same "uniform dimension".

14.8 (Aharoni [1]). *There is a constant K so that for every separable metric space (X, d) there is a map $T: X \to c_0$ satisfying the condition $d(x, y) \leqslant \|Tx - Ty\| \leqslant Kd(x, y)$ for every $x, y \in X$. Hence every separable Banach space is uniformly homeomorphic to a bounded subset of c_0.*

14.9 (Aharoni [2]). *For $1 \leqslant p \leqslant 2$, $1 \leqslant q < \infty$, L^p is uniformly homeomorphic to a subset of l^q, i.e. there is a subset $Z \subset l^q$ and a homeomorphism $f: L^p \to Z$ such that f and f^{-1} are uniformly continuous. Moreover, L^p is uniformly homeomorphic to a bounded subset of itself.*

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