NOTE I.

On Haar's measure

by

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§ 1. This Note is devoted to the theory of measure due to Alfred Haar [1]. Haar's beautiful and important theory deals with measure in those locally compact separable spaces for which the notion of congruent sets is defined. His measure fulfils the usual conditions of ordinary Lebesgue measure: congruent sets are of equal measure and all Borel sets (more generally, all analytic sets) are measurable. The theory has important applications in that of continuous groups.

To complete the definitions of Chap. II, § 2, we shall say that a set situated in a metrical space is *compact*, if every infinite subset of the set in question has at least one point of accumulation. A metrical space is termed *locally compact* if each point of this space has a neighbourhood which is compact.

§ 2. In what follows we shall denote by $\boldsymbol{E}$ a fixed metrical space, separable and locally compact, and we shall suppose that, for the sets situated in $\boldsymbol{E}$, the notion of *congruence* $\cong$ is defined so as to fulfil the following conditions:

i_1. $A \cong B$ *implies* $B \cong A$; $A \cong B$ *and* $B \cong C$ *imply* $A \cong C$;

i_2. *If* A *is a compact open set and* $A \cong B$, *then the set* B *is itself open and compact*;

i_3. *If* $A \cong B$ *and* $\{A_n\}$ *is a (finite or infinite) sequence of open compact sets such that* $A \subset \sum_n A_n$, *then there exists a sequence of sets* $\{B_n\}$ *such that* $B \subset \sum_n B_n$ *and such that* $A_n \cong B_n$ *for* $n = 1, 2, \ldots$;

i_4. *Whatever be the compact open set* A, *the class of the sets congruent to* A *covers the whole space* $\boldsymbol{E}$;

i_5. *If* $\{S_n\}$ *is a sequence of compact concentric spheres with radii tending to* 0, *and* $\{G_n\}$ *is a sequence of sets such that* $G_n \cong S_n$, *then the relations* $a = \lim_n a_n$ *and* $b = \lim_n b_n$, *where* $a_n \in G_n$ *and* $b_n \in G_n$, *imply* $a = b$.

§3. Given two compact open sets A and B, the class of the sets congruent to A covers, by i_4, the set $\bar{B}$. It therefore follows from the theorem of Borel-Lebesgue that there exists a finite system of sets congruent to A which covers $\bar{B}$. Let $\mathrm{h}(B,A)$ denote the least number of sets which constitute such a system.

It is easy to show by means of i_1—i_5 that, for any three compact open sets A, B and C, the following propositions are valid:

ii_1. $C\subset B$ *implies* $\mathrm{h}(C,A)\leqslant\mathrm{h}(B,A)$;

ii_2. $\mathrm{h}(B+C,A)\leqslant\mathrm{h}(B,A)+\mathrm{h}(C,A)$;

ii_3. $B\cong C$ *implies* $\mathrm{h}(B,A)=\mathrm{h}(C,A)$;

ii_4. $\mathrm{h}(B,A)\leqslant\mathrm{h}(B,C)\cdot\mathrm{h}(C,A)$;

ii_5. If $\varrho(A,B)>0$ *and if* $\{S_n\}$ *is a sequence of compact concentric spheres with radii tending to* 0, *then there exists a positive integer* N *such that, for every* $n>N$,

$$\mathrm{h}(A+B,S_n)=\mathrm{h}(A,S_n)+\mathrm{h}(B,S_n). \tag{3.1}$$

All these propositions are obvious, except perhaps ii_5. To prove the latter, let us suppose, if possible, that there exists an increasing sequence of positive integers $\{n_i\}$ such that (3.1) does not hold for any of the values $n=n_i$. There would then exist a sequence of sets $\{G_i\}$ such that $G_i\cong S_{n_i}$ while $A\cdot G_i\neq 0$ and $B\cdot G_i\neq 0$. Consider now arbitrary points $a_i\in A\cdot G_i$ and $b_i\in B\cdot G_i$. Since the sets A and B are compact, the sequences $\{a_i\}$ and $\{b_i\}$ contain respectively convergent subsequences $\{a_{i_j}\}$ and $\{b_{i_j}\}$. Let $a=\lim\limits_j a_{i_j}$ and $b=\lim\limits_j b_{i_j}$. By i_5 we must have $a=b$, and this is impossible since, by hypothesis, $\varrho(A,B)=0$.

We shall now suppose given a fixed compact open set G and a sequence $\{S_n\}$ of concentric spheres, with radii tending to 0, which are situated in G and therefore clearly compact. For every compact open set A, we write

$$l_n(A)=\frac{\mathrm{h}(A,S_n)}{\mathrm{h}(G,S_n)}.$$

We then have, by ii_4,

$$\mathrm{h}(A,S_n)\leqslant\mathrm{h}(A,G)\cdot\mathrm{h}(G,S_n)\quad\text{and}\quad\mathrm{h}(G,S_n)\leqslant\mathrm{h}(G,A)\cdot\mathrm{h}(A,S_n),$$

and hence for each $n=1,2,\ldots$, $1/\mathrm{h}(G,A)\leqslant l_n(A)\leqslant\mathrm{h}(A,G)$.

Thus, $\{l_n(A)\}$ *is a bounded sequence whose terms exceed a fixed positive number.*

§ 4. We now make use of the following theorem (cf. S. Banach [I, p. 34] and S. Mazur [1]), in which $\{\xi_n\}$ and $\{\eta_n\}$ denote arbitrary bounded sequences of real numbers, a and b denote real numbers, and the symbols lim, lim sup and lim inf have their usual meaning:

To every bounded sequence $\{\xi_n\}$ we can make correspond a number $\underset{n}{Lim}\,\xi_n$, termed generalized limit, in such a manner as to fulfil the following conditions:

1) $\underset{n}{Lim}\,(a\xi_n+b\eta_n)=a\cdot\underset{n}{Lim}\,\xi_n+b\cdot\underset{n}{Lim}\,\eta_n$,

2) $\liminf\limits_{n}\xi_n\leqslant\underset{n}{Lim}\,\xi_n\leqslant\limsup\limits_{n}\xi_n$,

3) $\underset{n}{Lim}\,\xi_{n+1}=\underset{n}{Lim}\,\xi_n$.

The last condition implies that *the generalized limit remains unaltered, when we remove from a sequence a finite number of its terms.*

Let us now write, for every compact open set A,

$$l(A)=\underset{n}{Lim}\,l_n(A). \tag{4.1}$$

We then have, for any compact open sets A and B:

iii$_1$. $0<l(A)<+\infty$;

iii$_2$. $A\subset B$ *implies* $l(A)\leqslant l(B)$;

iii$_3$. $A\cong B$ *implies* $l(A)=l(B)$;

iii$_4$. $l(A+B)\leqslant l(A)+l(B)$;

iii$_5$. $\varrho(A,B)>0$ *implies* $l(A+B)=l(A)+l(B)$.

§ 5. This being so, we denote, for an arbitrary set $X\subset \boldsymbol{E}$, by $\Gamma(X)$ the lower bound of all the numbers $\sum\limits_{n} l(A_n)$ where $\{A_n\}$ is any sequence of compact open sets such that $X\subset\sum\limits_{n}A_n$. We shall show that the function of a set Γ, thus defined, fulfils the following conditions:

1^0 *We have always* $0\leqslant\Gamma(X)$ *and there exist sets* X *for which we have* $0<\Gamma(X)<+\infty$; *this is, in particular, the case of all compact open sets* X;

2^0 $X_1\subset X_2$ *implies* $\Gamma(X_1)\leqslant\Gamma(X_2)$;

3^0 $X\subset\sum\limits_{n}X_n$ *implies* $\Gamma(X)\leqslant\sum\limits_{n}\Gamma(X_n)$;

4^0 $\varrho(X_1,X_2)>0$ *implies* $\Gamma(X_1+X_2)=\Gamma(X_1)+\Gamma(X_2)$;

5^0 $X_1\cong X_2$ *implies* $\Gamma(X_1)=\Gamma(X_2)$.

Proof. 1^0. Let X be a compact open set. We have, by definition, $\Gamma(X) \leqslant l(X) < +\infty$.

On the other hand, there clearly exists for each $\varepsilon > 0$ a (finite or infinite) sequence of compact open sets $\{A_n\}$ such that $X \subset \sum_n A_n$ and $\Gamma(X) + \varepsilon \geqslant \sum_n l(A_n)$. Let S be any sphere contained in X. Since the set $\bar{S}$ is closed and compact, this set, and *a fortiori* the set S, is already covered by a finite subsequence $\{A_{n_i}\}$ of $\{A_n\}$. In view of iii_2 and iii_4, we thus have

$$l(S) \leqslant l(\sum_i A_{n_i}) \leqslant \sum_i l(A_{n_i}) \leqslant \sum_n l(A_n) \leqslant \Gamma(X) + \varepsilon.$$

Hence, ε being arbitrary, it follows that $l(S) \leqslant \Gamma(X)$, and finally, by iii_1, that $0 < \Gamma(X)$.

2^0 and 3^0 are obvious.

4^0. $\varrho(X_1, X_2) > 0$ implies that there exist two open sets G_1 and G_2 such that $X_1 \subset G_1$, $X_2 \subset G_2$ and $\varrho(G_1, G_2) > 0$. On the other hand, there exists for each $\varepsilon > 0$ a sequence of compact open sets $\{A_n\}$ such that

$$X_1 + X_2 \subset \sum_n A_n \quad \textit{and} \quad \Gamma(X_1 + X_2) + \varepsilon \geqslant \sum_n l(A_n). \tag{5.1}$$

Write $A_n^{(1)} = A_n \cdot G_1$ and $A_n^{(2)} = A_n \cdot G_2$. Since the sets $A_n^{(1)}$ and $A_n^{(2)}$ are open and compact, and since their distance, like that of G_1 and G_2, is positive, we have, on account of iii_5 and iii_2,

$$l(A_n^{(1)}) + l(A_n^{(2)}) = l(A_n^{(1)} + A_n^{(2)}) \leqslant l(A_n). \tag{5.2}$$

But, since on the other hand $X_1 \subset \sum_n A_n^{(1)}$ and $X_2 \subset \sum_n A_n^{(2)}$, we have the inequalities $\Gamma(X_1) \leqslant \sum_n l(A_n^{(1)})$ and $\Gamma(X_2) \leqslant \sum_n l(A_n^{(2)})$, so that, by (5.2), $\Gamma(X_1) + \Gamma(X_2) \leqslant \sum_n l(A_n)$. Hence by (5.1), ε being arbitrary, we obtain $\Gamma(X_1) + \Gamma(X_2) \leqslant \Gamma(X_1 + X_2)$, and finally by 3^0, $\Gamma(X_1) + \Gamma(X_2) = \Gamma(X_1 + X_2)$.

5^0 follows at once from i_3 and iii_3.

§ 6. It follows from the properties 1^0—4^0 of the function Γ that the latter is an outer measure in the sense of Carathéodory (cf. Chap. III, § 4) and therefore determines in $\boldsymbol{E}$ a class of sets measurable $(\mathfrak{L}_\Gamma)$, that we shall call, simply, *measurable sets*. We see at once that for each set X in $\boldsymbol{E}$ the number $\Gamma(X)$ is the lower bound of the measures (Γ) of the open sets containing X. It follows, in particular, that Γ is a regular outer measure (cf. Chap. II, § 6).

Finally, since the space $\boldsymbol{E}$ can be covered by a sequence of measurable sets of finite measure (e. g. by a sequence of compact spheres), we easily establish, for the measure Γ, conditions of measurability ($\mathfrak{L}_I$) similar to those of Theorem 6.6, Chap. III. In particular, we shall have:

(6.1) *In order that a set E be measurable, it is necessary and sufficient that there exist a set* ($\mathfrak{G}_\delta$) *containing E and differing from it by at most a set of measure zero.*

§ 7. We conclude this note by giving two examples of spaces $\boldsymbol{E}$ with the notion of congruence subject to the conditions of § 2.

Example 1. Let $\boldsymbol{E}$ be a metrical space which is separable and locally compact, and suppose that, among the one to one transformations, continuous both ways, by which the whole space $\boldsymbol{E}$ is transformed into the whole space $\boldsymbol{E}$, there exists a class $\mathfrak{M}$ of transformations subject to the conditions:

1) $T \in \mathfrak{M}$ *implies* $T^{-1} \in \mathfrak{M}$;

2) *If* $T_1 \in \mathfrak{M}$ *and* $T_2 \in \mathfrak{M}$, *then* $T_1 T_2 \in \mathfrak{M}$;

3) *For every pair a, b of points of E, there exists a transformation* $T \in \mathfrak{M}$ *such that* $T(a) = b$;

4) *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences of points of* $\boldsymbol{E}$ *such that* $\lim_n a_n = \lim_n b_n$, *and if* $\{T_n\}$ *is a sequence of transformations belonging to* $\mathfrak{M}$ *such that the sequences* $\{T_n(a_n)\}$ *and* $\{T_n(b_n)\}$ *are convergent also, then we have* $\lim_n T_n(a_n) = \lim_n T_n(b_n)$.

Two sets $A \subset \boldsymbol{E}$ and $B \subset \boldsymbol{E}$ will be termed *congruent*, if there exists a transformation $T \in \mathfrak{M}$ such that $T(A) = B$ (where $T(A)$ denotes the set into which A is transformed, i. e. the set of all the points $T(a)$ for which $a \in A$).

It is easy to verify that the conditions i_1—i_5 are fulfilled.

As special cases of such spaces $\boldsymbol{E}$ we may mention: Euclidean n-dimensional space with $\mathfrak{M}$ interpreted as the class of all translations and rotations; the 3-dimensional sphere with $\mathfrak{M}$ interpreted as the class of all rotations.

Let us observe that, in the space considered, the sets which are congruent to open sets are themselves open. On the other hand, on account of 5^0, p. 316, the sets congruent to sets of measure (Γ) zero are themselves of measure zero. It follows therefore from (6.1) that in the space $\boldsymbol{E}$ considered *the sets which are congruent to measurable sets, are themselves measurable.*

Example 2. Suppose that a metrical space $\boldsymbol{E}$, separable and locally compact, constitutes a group, i. e. that with each pair a, b of elements of $\boldsymbol{E}$ there is associated an element ab of $\boldsymbol{E}$, called *product,* in such a manner that the following conditions are fulfilled:

1) $(ab)c = a(bc)$ *(whatever be the elements* a, b *and* c *of* $\boldsymbol{E}$*)*;

2) *there exists in* $\boldsymbol{E}$ *a unit-element* 1 *such that we have* $1 \cdot a = a \cdot 1 = a$ *for every* $a \in E$;

3) *to each element* $a \in \boldsymbol{E}$ *there corresponds an inverse element* $a^{-1} \in \boldsymbol{E}$ *which fulfils the equation* $aa^{-1} = 1$.

Suppose further that $\boldsymbol{E}$ fulfils the conditions:

4) *if* $\lim\limits_n a_n = a$ *and* $\lim\limits_n b_n = b$, *then* $\lim\limits_n a_n b_n = ab$;

5) *if* $\lim\limits_n a_n = a$, *then* $\lim\limits_n a_n^{-1} = a^{-1}$.

Given any element $c \in \boldsymbol{E}$ and any set $B \subset \boldsymbol{E}$, we denote by cB the set of all the elements $a \in \boldsymbol{E}$ such that $a = cb$ where $b \in B$.

Given an element a of $\boldsymbol{E}$, we write, for every element $x \in \boldsymbol{E}$, $T_a(x) = ax$. Thus each element a of $\boldsymbol{E}$ determines a transformation T_a, clearly one to one and continuous both ways, of the space $\boldsymbol{E}$ into itself. Denoting the class of all these transformations by $\mathfrak{M}$, we see at once that the conditions 1)—4), p. 318, are fulfilled. In accordance with the definition of congruence employed in Example 1, two sets A and B in the space in question are congruent if there exists an element c such that $B = cA$.